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The harmonic measures of Lucy Garnett

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Abstract

This paper presents an improved approach to the theory of harmonic measures for foliated spaces introduced by Garnett. This approach is based on a method for solving elliptic equations on foliated spaces and on the Hille–Yosida theory. The diffusion semigroup of a general Laplacian and its infinitesimal generator are made explicit. Applications of the path space to the dynamical study of a foliated space are described. In particular, the final section studies cocycles on foliated spaces, a formula for their asymptotic limit, and some analytic and geometric consequences.

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0. Introduction

The ergodic theory of foliations is not quite as developed as that of flows, perhaps because foliations which have invariant measures are rather scarce. An important development in this area was the paper of Garnett [14], where she introduced harmonic measures for a foliation, proved that they always exist, and exhibited the basic facts of ergodic theory with respect to these measures.

The local structure of these harmonic measures is quite similar to that of the usual invariant measures. Locally, they decompose into a measure on a transversal and

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measures on the plaques. These plaque measures are not only smooth, but the product of a positive harmonic function times the Riemannian volume of the plaque.

The idea behind the harmonic measures is that one solves the problem of going to infinity along the leaves by introducing a semigroup of operators indexed by the positive real numbers. From this point of view it is then possible to formulate an ergodic theorem for foliated spaces.

This paper revisits Garnett's. In that paper, Garnett introduced the semigroup of operators leaf by leaf, using the heat kernel on each leaf. The main point was then to show that this process takes continuous functions on the foliated space to continuous ones. She gives an argument that verifies this claim when the foliated space has no holonomy. However, there are serious difficulties in extending it when there is holonomy.

One of the purposes of the present paper is to describe a different approach to the continuity of the diffusion semigroup attached to a Laplace operator along the leaves of a foliated space. The proof here uses the Hille–Yosida theorem on semigroups of operators, and it works regardless of whether the foliated space has holonomy or not. This method has the advantage of avoiding the use of heat kernels (and heat kernel estimates) along the leaves, and makes it possible to treat general Laplacians, not necessarily those attached to a metric tensor along the leaves. Two consequences of this approach are a general method for solving elliptic equations on foliated spaces, and a construction of the heat kernel on the leaves associated to any elliptic operator. Harmonic measures associated to Laplacians with non-trivial drift are actually important (the so called Patterson–Sullivan measures from the theory of Kleinian groups are of this kind) and a treatment along the lines of [14], even in a foliated space without holonomy, seems out of the question.

The semigroups of diffusion operators associated to a Laplacians along the leaves is an integrated version of a semiflow taking place on the space of paths along leaves. On each leaf, it reduces to Brownian motion. This is implicit in Garnett's paper, but I thought useful to make it explicit here, and develop a global construction. A justification for this is that this space provides a very useful tool for studying the asymptotic behavior of foliated spaces, an excellent example being Ghys [16]. This is then used to work out an ergodic decomposition of harmonic measures for a foliated space in the spirit of the original approach of Kryloff and Bogoliouboff.

The final section is devoted to the study of asymptotic properties of cocycles on a foliated space. Such cocycles arise for example from one-forms on the space. One example of such analysis is the theorem that the von Neumann algebra of a harmonic (but not invariant) measure is type III. Another example is the existence of resilient leaves in the presence of some quasi-conformal transverse structure, a theorem in the spirit of one of the well-known unpublished theorem of Duminy. Yet another example is the question of uniformization of surface laminations.

1. Differential equations on manifolds

Let D be a domain in $\mathbb{R}^d$ such that $\bar{D}$ is a manifold with boundary of class C^k. For $r<k$, let $C^r(\bar{D})$ denote the space of functions $f: D \to \mathbb{R}$ which are of class C^r on D, and

whose derivatives extend continuously to ∂D. The norm of an element f of $C^r(\bar{D})$ is

$$||f||_r = \sum_{i=0}^{r} \sum_{|m|=i} \sup_D |\partial^m f|,$$

where $\partial^m = \partial^{|m|}/\partial x_{s_1} \cdots \partial x_{s_r}$ is the multi-derivative corresponding to the multi-index $m = (s_1, \ldots, s_r)$, with $|m| = s_1 + \cdots + s_r$.

The symbol $C^{r,\alpha}(\bar{D})$, where $r \geqslant 0$ and $0 < \alpha \leqslant 1$, denotes the space of functions $f \in C^r(\bar{D})$ whose r-derivatives are α-Hölder continuous functions on D. It becomes a Banach space once it is endowed with the norm

$$||f||_{r,\alpha} = ||f||_r + \sum_{|m|=r} ||\partial^m f||_\alpha.$$

Here $||f||_\alpha$ denotes the α-Hölder norm of the function f on D. It is convenient to write $C_{r,0}(\bar{D})$ for $C_r(\bar{D})$. If $\alpha \leqslant \beta$, then the canonical inclusion of $C^{r,\beta}(\bar{D})$ into $C^{r,\alpha}(\bar{D})$ is norm continuous. More generally,

Proposition 1.1. *Let D be a regular domain. If $r + \alpha > r' + \alpha'$, then the inclusion $C^{r,\alpha}(\bar{D}) \subset C^{r',\alpha'}(\bar{D})$ is well defined and continuous.*

Let L be a manifold of dimension d and class C^k. Let D be a bounded regular domain in L, that is to say, D is a connected open set such that $\bar{D}$ is a C^k-submanifold with boundary. The Banach spaces $C^{r,\alpha}(\bar{D})$, $r < k$, are constructed as follows. Let (U_i, ϕ_i) be a cover of $\bar{D}$ by manifold charts, where each $\phi_i^{-1}(U_i)$ is Euclidean space or half-space, and such that the image of the Euclidean balls and half balls under the ϕ_i still cover $\bar{D}$. To carry the classical estimates from domains in Euclidean space to domains in L, it is convenient to assume that the half balls mentioned here are C^k-smooth, something that can be achieved in the obvious way. The space $C^{r,\alpha}(\bar{D})$ consist of all functions $f : D \to \mathbb{R}$ which are r-times continuously differentiable and with α-Hölder derivatives of order r, all of then extending continuously to $\bar{D}$. The norm of $f \in C^{r,\alpha}(\bar{D})$ is

$$||f||_{r,\alpha} = \sum ||f|_{V_i}||_{r,\alpha},$$

where V_i are the Euclidean balls and half balls from the atlas.

A function f on L is locally Hölder if there is a family of manifolds charts (U, ϕ) covering L such that the function $f \circ \phi$ is α-Hölder on U, for some $\alpha \in (0, 1)$. If D is a bounded domain in L, and if f is a locally Hölder function on L, then f belongs to $C^{0,\alpha}(\bar{D})$ for some α.

A Laplace operator on a manifold L, of class C^k, is a linear operator acting on functions, which is locally of the form

$$\Delta f(x) = \sum_{ij} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i} f(x),$$

where the coefficients a_{ij} and b_i are Hölder and the matrix function (a_{ij}) is symmetric and positive definite.

A very useful property of Laplace operators is the following basic fact, which is a well-known calculus lemma.

Lemma 1.2. *Let Δ be a Laplace operator and let $f : D \to \mathbb{R}$ be of class C^2 on the domain D. If $x \in D$ is a local minimum, then $\Delta f(x) \geqslant 0$, while if $x \in D$ is a local maximum, then $\Delta f(x) \leqslant 0$.*

Laplace operators are of local nature, which has the following consequence.

Lemma 1.3. *Let $\pi : L' \to L$ be a cover of L. If Δ is an operator on L, there is a unique operator Δ' on L' such that $\Delta(f \circ \pi) = (\Delta' f) \circ \pi$.*

The purpose of the rest of this section is to discuss the global Dirichlet problem on a manifold L with a Laplace operator Δ. This problem requests the following: given a bounded function g on L and $\lambda > 0$, find a bounded function f which satisfies

$$\lambda f - \Delta f = g$$

on L. Such solution f will be constructed as a limit of solutions to Dirichlet problems on a family of bounded regular domains which expand to L.

The following statement is the solution to the Dirichlet problem on a bounded regular domain in a manifold. The proof follows from the similar statement on domains in Euclidean space. References for this material are [18,28].

Theorem 1.4. *Let D be bounded regular domain in L, of class C^k, $k > 2$. Let Δ be a Laplace operator on L. Given a function $g \in C^{r-2,\alpha}(\bar{D})$ (with $2 \leqslant r < k$) and $\varphi \in C^{r,\alpha}(\partial D)$, there is a unique function $f \in C^{r,\alpha}(\bar{D})$ which solves the problem*

$$(D, g, \varphi, \lambda) = \begin{cases} \lambda f - \Delta f = g & \text{on } D, \\ f = \varphi & \text{on } \partial D. \end{cases}$$

There are two main issues involved in this theorem. One is the question of uniqueness, which will appear below for related reasons, and the other is that of existence, which involves the so-called Schauder global estimates. These are the content of the next statement.

Theorem 1.5. *Let L, D and Δ be as above. There is a constant K, which depends on the operator Δ and the domain D, such that*

$$||f||_{r,\alpha} \leqslant K(||f||_{0,\alpha} + ||g||_{r-2,\alpha} + ||\varphi||_{r,\alpha}).$$

Also of relevance for the purposes of this paper is the following compactness result; a proof is to be found in the same references.

Theorem 1.6. *Let L, D and Δ be as above. If S is a bounded set in $C^{r,\alpha}(\bar{D})$, then S is precompact in $C^{r,\beta}(\bar{D})$, for $\beta<\alpha$.*

The problem of existence of solutions to the Dirichlet problem on a manifold will now be discussed. Let L be a manifold and Δ a Laplacian on L. Let $g: L \to \mathbb{R}$ be a continuous function on L, and let $\lambda > 0$. If D is a bounded regular domain in L and κ is a constant function on ∂D, then there exists a function $f: \bar{D} \to \mathbb{R}$ which solves the problem (D, g, κ, λ) described in Theorem 1.4, at least under the assumption that g is locally Hölder.

Lemma 1.7. *Let g be a continuous function on L, let $\lambda > 0$, and suppose that f is a continuous function on $\bar{D}$ such that $\lambda f - \Delta f = g$ on D. Then*

(1) *If the constant function κ is such that $g \geqslant \lambda\kappa$ on D, and $f \geqslant \kappa$ on ∂D, then $f \geqslant \kappa$ on D.*
(2) *If $g \leqslant \lambda\kappa$ on D and $f \leqslant \kappa$ on ∂D, then $f \leqslant \kappa$ on D.*

Proof. If the minimum of f occurs on ∂D, then there is nothing to prove. If $x \in D$ is a minimum for f, then $\Delta f(x) \geqslant 0$, hence $\lambda f(x) = g(x) + \Delta f(x) \geqslant \lambda\kappa$. The other statement uses that $\Delta f \leqslant 0$ at a maximum. $\square$

In particular, if $g \geqslant 0$ and $\kappa = 0$, the solution f_D is non-negative on D. Moreover,

Lemma 1.8. *Let f, f' be two continuous functions on $\bar{D}$, both of them solutions to the equation $\lambda f - \Delta f = g$ on D. If $f' \geqslant f$ on ∂D, then $f' \geqslant f$ on D.*

Proof. The function $f' - f$ is continuous on $\bar{D}$ and $\geqslant 0$ on ∂D. It satisfies

$$\lambda(f' - f) = \Delta(f' - f).$$

At a minimum, if it occurs on D, it must hold that $\Delta(f' - f) \geqslant 0$, and from the equation it follows that $f' - f \geqslant 0$ at this minimum. Hence $f' \geqslant f$ on D. $\square$

Corollary 1.9. *There exists at most one solution to the Dirichlet problem (D, g, κ, λ).*

Lemma 1.10. *Let f be a solution to the Dirichlet problem (D, g, κ, λ). Suppose that $g \geqslant \lambda\kappa$ on D. Then $\lambda f \leqslant \sup_D g$ on D.*

Proof. Since $f \equiv \kappa$ on ∂D, Lemma 1.7 yields $f \geqslant \kappa$. Hence $\lambda f \geqslant \inf_D g$. Let x be a maximum for f on $\bar{D}$. If $x \in \partial D$, then $f \leqslant \kappa$ on D, and the conclusion follows. If $x \in D$, then $\Delta f(x) \leqslant 0$; hence $\lambda f(x) = g(x) + \Delta f(x) \leqslant \sup_D g$. $\square$

Corollary 1.11. *If* $\inf_D g \geqslant \lambda\kappa$, *then the solution* f *above satisfies* $\lambda\|f\| \leqslant \|g\|$ *on* D.

Proposition 1.12. *Let* g *be a bounded, locally Hölder, continuous function on* L. *Then there exists a bounded solution* f *to the equation* $\lambda f - \Delta f = g$ *on* L. *This solution* f *is defined by*

$$f = \sup_{D \subset L} f_D,$$

where D *is a relatively compact domain in* L, *and* f_D *is the solution to the Dirichlet problem* (D, g, κ, λ) *with* $g \geqslant \lambda\kappa$. *The norm of this solution is* $\lambda\|f\| \leqslant \|g\|$.

Proof. If $D \subset D'$, then the corresponding solutions $f_{D'}$ and f_D satisfy $f_{D'} - f_D \geqslant 0$ on ∂D, hence on D. Thus the family of functions f_D increases with respect to the ordering of the domains. By Lemma 1.10, the elements of this family are bounded above in terms of g; hence the supremum f exists. If D_n is an increasing sequence of bounded regular domains exhausting L, then the family of corresponding functions f_n is precompact in each space $C^{r,\beta}(\bar{D})$ associated to a regular domain in L. Therefore, the sequence converges locally uniformly to f, thus $f \in C^{r,\beta}(\bar{D})$. Since each f_n solves $\lambda f_n - \Delta f_n = g$, so does f. $\square$

Without further restrictions on the operator Δ, the solution f just constructed may not be unique. The method of construction has a dual version which produces another solution.

Proposition 1.13. *For each bounded domain* D *in* L, *let* f'_D *be the solution to* $\lambda f - \Delta f = g$ *on* D *with boundary conditions* $f = (1/\lambda)\|g\|$ *on* ∂D. *Then*

$$f = \inf_D f'_D$$

is a solution to $\lambda f - \Delta f = g$ *on* L *with norm* $\|f\| \leqslant (1/\lambda)\|g\|$.

A brief discussion of the parabolic differential equation, also called heat equation, associated to an elliptic operator (of second order) needs to be included here.

Let Δ be an elliptic operator on a manifold L. A function $f(x,t)$ on $L \times (0,\infty)$ satisfies the heat equation with initial conditions $f(x)$ if

$$\frac{d}{dt} f - \Delta f = 0$$

and $f(x,t) \to f(x)$ as $t \to 0$. This last limit can be taken in several spaces, depending on what hypotheses are imposed on f.

A function $p(x,y;t)$ is a fundamental solution to the heat equation if

$$\frac{d}{dt} p(x,y;t) = \Delta_x p(x,y;t),$$

and for every function f with compact support, the function

$$(D_t f)(x) = \int_L f(y) p(x, y; t) \cdot \mathrm{vol}(y)$$

satisfies the heat equation with f as initial conditions (meaning that $D_t f \to f$ as $t \to 0$ in an appropriate sense).

It follows from the maximum principle for parabolic differential equations that the function $p(x, y; t)$ is strictly positive for $t > 0$. The reference Chavel [6] contains plenty of material on the properties of the heat kernel corresponding to the Laplace operator of a complete metric tensor on a manifold. All the theory that will be required for general heat kernels follows from those of Riemann metrics. Indeed, if Δ is an elliptic operator with heat kernel $p(x, y; t)$, then the operator $(\Delta + \Delta^*)/2$ is the Laplacian of a metric plus a potential, and its heat kernel is

$$\tfrac{1}{2}(p(x, y; t) + p(y, x; t)).$$

Observe also that the operators Δ and Δ^* have the same principal part, that is to say, their corresponding metric tensor is the same, and so is that of the symmetrized operator.

From the theory of heat kernel on a manifold, the following "normal" (or Gaussian) estimate, due to Cheng-Li-Yau [7], will be used, specially in the final section.

Theorem 1.14. *Let L be a complete, non-compact, Riemannian manifold of bounded geometry. Let $p(x, y; t)$ denote the heat kernel associated to the metric Laplacian of L. Then, given $T > 0$, there is a constant B which depends on the geometry of L and on T, such that the following upper bound holds:*

$$p(x, y; t) \leqslant \frac{B}{t^{d/2}} \exp(-d(x, y)^2/16t),$$

for all $0 < t \leqslant T$.

This estimate can be extended to cover the symmetrized operator $(\Delta + \Delta^*)/2$, at least when working on a compact foliated space. The idea is to obtain such estimate from the heat kernel of the metric tensor Laplacian and the Feynman–Kac formula for Schrödinger operators. For bounded times, the upper bound has the same formal expression.

2. Fundamentals of foliated spaces

The definition of foliated spaces requires to introduce first the definition of smooth function. Let Z be a separable, metrizable space, and let U be an open subset of the product $\mathbb{R}^d \times Z$. A function $f : U \to \mathbb{R}$ is said to be differentiable of class C^r at a point $(x_0, z_0) \in U$ if there is a neighborhood of this point of the form $D \times Z'$ such that the function $z \mapsto f(\cdot, z) \in C^r(D)$ is continuous on Z, where $C^r(D)$ has the topology of uniform convergence of all derivatives of order $\leqslant r$. The function f is said to be of class C^r in U if it is differentiable of class C^r at every point of U.

A function $f : U \subset \mathbb{R}^d \times Z \to \mathbb{R}$ is locally Hölder if each point has a neighborhood of the form $D \times Z'$, with $\bar{D} \times Z' \subset U$, such that the induced map $Z' \to C^{r,\alpha}(\bar{D})$ is continuous.

Generalizing this definition, a map $f : D \times Z \to D' \times Z'$ is said to be a map of foliated spaces if the image of each plaque $D \times \{z\}$ is contained in a plaque $D' \times \{z'\}$. Such map f is of class $C^{r,\alpha}$ at a point (x, z) if there is a neighborhood of this point of the form $V \times Y$ such that the induced map $z \in Z \mapsto f(\cdot, z) \in C^{r,\alpha}(V, D')$ is continuous.

Definition 2.1. A locally compact, separable and metrizable space M is said to be a d-dimensional foliated space of class C^k if it admits a cover by open sets U_i (called flow boxes or charts) and homeomorphisms

$$\varphi_i : U_i \to D_i \times Z_i,$$

where D_i is an open disc in $\mathbb{R}^d$, and such that the overlap maps $\varphi_j \varphi_i^{-1}$ are of the form

$$\varphi_j \varphi_i^{-1}(x_i, y_i) = (x_i(x_j, y_j), z_i(z_j)),$$

where each map

$$\varphi_i(U_i \cap U_j) \subset D_i \times Z_i \to D_j \times Z_j$$

is smooth of class C^k.

Let $C(M)$ denote the space of bounded continuous functions on M, which is a Banach space when endowed with the supremum norm. The space of smooth functions on M of class C^r is denoted by $C^r(M)$; it is a Frechet space with the topology of uniform convergence on compact sets of all leaf derivatives of all orders $\leqslant r$. The subspace consisting of functions with compact support is denoted by $C^r_c(M)$. Other basic facts that will be used are the existence of smooth partitions of unity and the fact that a compact M can be embedded in a real Hilbert space.

Theorem 2.2. *Let M be a foliated space of class C^r. Then there is a C^r-embedding of M into a real separable Hilbert space $\mathbb{E}$.*

Proof. See [5]. The standard technique for embedding a manifold in Euclidean space works in this situation, once the following facts are noted. First, M admits partitions of unity of class C^k; second, M admits a countable cover by open sets of the form $D \times T$, where T is a locally compact metrizable space; third, a compact metrizable space admits a topological embedding into separable real Hilbert space; and fourth, a separable Hilbert space $\mathbb{E}$ of infinite dimension is diffeomorphic to a countable product $\mathbb{E} \times \mathbb{E} \times \cdots$. $\square$

Each leaf L of M is then a submanifold of $\mathbb{E}$. Let $N(L)$ denote its normal bundle with projection $\pi_L : N(L) \to L$. Then there is a neighborhood W of the zero section of $N(L)$ such that the inclusion $L \hookrightarrow \mathbb{E}$ extends to a local diffeomorphism $W \to \mathbb{E}$. Therefore, there is a foliated space in W which is the pullback of the foliated space M in $\mathbb{E}$. The leaf L can be identified with the zero section of $N(L)$, and therefore with a leaf of W. The leaves of this foliated space W are transverse to the fibers of π_L, and if L' is a leaf of W, then the map $\pi_L : L' \to L$ is locally a diffeomorphism.

If $f : \tilde{L} \to L$ is a covering of L, then the same reasoning can be applied to the pullback $f^*N(L)$ of the normal bundle to $\tilde{L}$. In particular, if $\tilde{L}$ is the holonomy cover of L, then the leaf $\tilde{L}$ of the neighborhood $\tilde{W}$ of the zero section of $f^*N(L)$ has no holonomy. Summarizing,

Corollary 2.3. *Let M be a foliated space of class C^2 embedded in $\mathbb{E}$ as a closed subspace. Let $L \subset M$ be a leaf of M and let L' be a covering of L. Then there exists a vector bundle N over L' with fiber $\mathbb{E}$ and a neighborhood W of the zero section of $N \to L'$ such that*:

(1) *For each $x \in L'$, the inverse image of x in W is diffeomorphic to the unit ball in $\mathbb{E}$. If M is compact, then W contains a ball of a definite radius on each fiber.*
(2) *The map $L' \to L \subset M$ extends to a local diffeomorphism $f : W \to \mathbb{E}$.*

The local diffeomorphism of (2) *permits to construct a foliated space $F \subset W$, $F = f^{-1}(M \cap f(W))$, with leaves of the same dimension as those of M and having the zero section L' as a leaf. The following properties also hold*:

(3) *The leaves of the foliated space F are transverse to the fibers of $N \to L'$.*
(4) *The projection $N \to L'$ restrict to a submersion of each leaf of F into L'.*
(5) *If the induced homomorphism $\pi_1 L' \to \pi_1 L$ has image contained in the kernel of the holonomy representation of L, then W and F can be chosen so that L' is without holonomy in F.*

Another useful consequence of Theorem 2.2 is the following.

Corollary 2.4. *Let M be as above embedded in $\mathbb{E}$ as a closed subset. Given a metric tensor along the leaves of M, there is a metric tensor on $\mathbb{E}$ which induces it. In particular, given a metric tensor along the leaves, inducing a distance d on each leaf, there is a distance function on M under which all leaf inclusions $L \hookrightarrow M$ are Lipschitz functions of constant $\leqslant 1$.*

Proof. The tangent space to a leaf of M is a finite-dimensional subspace of the tangent space to $\mathbb{E}$, hence it has an orthogonal complement. This metric is only continuous on $\mathbb{E}$, but that is sufficient for the conclusion. $\square$

More information on this basic structure and proofs of these basic facts can be found in [1,5].

3. Laplacians and harmonic measures

Let M be a foliated space of class C^k $(k \geqslant 2)$. An elliptic differential operator (of second order) A on M is a linear mapping of $C^r(M)$, $2 \leqslant r \leqslant k$, into $C(M)$ which can be expressed locally as follows. If $U \subset M$ is a chart of the form $U \cong D \times Z$, with coordinates $x = (x_1, \ldots, x_n)$ along the leaves, then

$$Af(x,z) = \sum_{ij} a_{ij}(x,z) \frac{\partial^2}{\partial x_i \partial x_j} f(x,z) + \sum_i b_i(x,z) \frac{\partial}{\partial x_i} f(x,z) + c(x,z) f(x,z),$$

where the matrix $(a_{ij}(x,z))$ is symmetric and positive definite. It is assumed that the matrix function $(a_{ij}(x,z))$ is of class at least C^2, and that the vector function $(b_i(x,z))$ is of class at least C^1. This differentiability hypothesis guarantees the existence of the formal adjoint operator A^*, which then implies that the following Green's formula holds.

Proposition 3.1. *Let D be a bounded regular domain on a leaf L of M. Then for functions f, h of class C^2 on L, one of them in $C^2_c(D)$, the following identity holds:*

$$\int_D f(A^* h) = \int_D h(Af).$$

It is straightforward to verify that an elliptic operator A can be written as

$$Af = \Delta f + X(f) + cf$$

where Δ is the Laplacian of the metric defined by the coefficients a_{ij}, the drift X is an operator of order one (a vector field along the leaves), and the potential c is an operator of order zero (a smooth function on M). The operator A is trivial on the constants if c is the constant function equal to 0, a property that can be verified in any local expression for A.

Definition 3.2. A Laplace operator (or Laplacian) on the foliated space M is an elliptic operator on M which vanishes on the constant functions.

The fundamental example of a Laplace operator on a foliated space M is the Laplacian of a metric tensor along the leaves. By virtue of Corollary 2.4, these always exist. A Laplacian will be usually denoted by the symbol Δ, but it is not meant to be the Laplacian associated to a metric tensor, unless it is made explicit or it is clear from the context. A Laplace operator Δ on M is then of the form

$\Delta = \Delta_0 + X$, where Δ_0 is the Laplacian of a metric tensor on M; its formal adjoint is $\Delta^* = \Delta_0 - X - \operatorname{div} X$.

The primary goal is to construct measures on M which are related to the Laplacian in the same way that a transverse invariant measure is related to first-order differential operators. The definition is the following.

Definition 3.3. Let M be a foliated space and let A be a differential operator. A measure m on M is said to be harmonic (with respect to A) if

$$\int_M Af(x) \cdot m(x) = 0$$

holds true for each compactly supported smooth function f on M.

The following lemma holds by viewing M as a disjoint union of manifolds.

Lemma 3.4. *Let M be a foliated space and let Δ be a Laplace operator on M. Let f be a function on M which is of class C^2 on each leaf. If f has a local maximum (respectively, local minimum) at a point x_0, then $\Delta f(x_0) \leqslant 0$ (respectively, $\geqslant 0$).*

The existence of harmonic measures on a compact foliated space is a consequence of this lemma and the Hahn–Banach theorem.

Theorem 3.5. *Let M be a compact foliated space, and let Δ be a Laplace operator on M. Then there is a probability measure m on M such that*

$$\int_M \Delta f(x) \cdot m(x) = 0$$

for every smooth function f on M.

Proof. By the Riesz representation theorem, a probability measure on M is the same as a continuous linear functional Λ on $C(M)$ of norm $||\Lambda|| = 1$, and such that $\Lambda(1) = 1$.

Thus, if Δ is a Laplace operator on the compact foliated space M, then a harmonic measure is just a positive linear functional on $C(M)$ which vanishes on functions of the form Δf. By the Hahn–Banach theorem, the existence of such linear functional is guaranteed if it can be shown that the closure of the range of Δ does not contain the constant functions. If it were otherwise, then there would be a sequence of functions f_n such that Δf_n converges uniformly to the constant function 1. That is, it must be that for some large n, $\Delta f_n(x) \geqslant 1/2$ for all x in M. But this contradicts the previous lemma, which says that Δf_n must vanish at some point of M. $\square$

Remark. I have known this little argument for a number of years, but have never seen it in print. The standard proofs use either a Cesaro limit argument or a fixed point theorem. Ghys tells me that it is folklore (see his paper [17]). Indeed, by chance I run across a paper of Weil [36, p. 114], where this argument based on extrema is used to construct measures invariant under certain groups of transformations of a compact space.

Example. This is a convenient place to expand on the remark about the Patterson–Sullivan measures made in the introduction. Let Γ be a (reasonable) group of transformations of hyperbolic space $\mathbb{H}^{n+1}$. The group Γ also acts on the sphere S^n and there it has a closed invariant set, the limit set Λ. Patterson (for $n=1$) and then Sullivan (in general), showed that there is a measure μ on S^n concentrated on Λ, and which transforms according to the rule $\gamma_*\mu = |\gamma'|^{-\delta}\mu$, where δ is the exponent of convergence of the Poincaré series of Γ.

Moreover, there is a function $h \in L^2(\mathbb{H}^{n+1}/\Gamma)$ which satisfies the functional equation $\Delta_0 h = \lambda h$, with $\lambda = \delta(\delta - n)$. This function admits an integral representation

$$h(x) = \int_{S^n} P(x,b)^{\delta} \cdot \mu(b),$$

where P is the Poisson kernel of $\mathbb{H}^{n+1}$.

There are two foliated spaces associated to Γ. One is the suspension $X = (\mathbb{H}^{n+1} \times \Lambda)/\Gamma$, the other is the manifold $N = \mathbb{H}^{n+1}/\Gamma$, a quotient of the first. Let Δ be the Laplace operator $\Delta f = (1/h)\Delta_0(hf) - \lambda f$. Its adjoint Δ^* satisfies $\Delta^* h^2 = 0$ and $\Delta^*(hP^{\delta}) = 0$. Then

$$\int_X f(x,b)h(x)P(x,b)^{\delta} \cdot \mathrm{vol}(x)\mu(b)$$

defines a Δ-harmonic measure on X which projects to a Δ-harmonic measure on N defined by

$$\int_N f(x)h^2(x) \cdot \mathrm{vol}(x).$$

4. Semigroups of operators

The proper development of harmonic measures requires some material from the theory of semigroups of operators on Banach spaces.

Definition 4.1. A one parameter family $\{D_t \mid t \geqslant 0\}$ of bounded linear operators on a Banach space E, with norm $|| \, ||$, is called a contraction semigroup of operators if it satisfies the following conditions:

(1) $D_{t+s} = D_t D_s$, for all $s, t \geqslant 0$,
(2) $\lim_{t\to 0} ||D_t f - f|| = 0$, for every $f \in E$, and
(3) The operator norm $||D_t|| \leqslant 1$.

It follows from the definition that the map $(t,f)\in[0,\infty)\times E\to D_t f\in E$ is continuous, and also that $\lim_{t\to s}||D_t f - D_s f|| = 0$ for all $f\in E$ and all $s\geqslant 0$.

Definition 4.2. The infinitesimal generator of a contraction semigroup D_t is defined by the formula

$$Af = \lim_{t\to 0}\frac{D_t f - f}{t}.$$

Its domain $\mathscr{D}_A$ consists of all elements f of E for which this limit exists in E.

The purpose of this section is to show that if Δ is a Laplace operator on a compact foliated space, then there is a contraction semigroup of operators on the Banach space $C(M)$ whose infinitesimal generator agrees with Δ in a dense subspace of $C(M)$. In this situation, the family $\{D_t\}$ is called the diffusion semigroup of Δ.

Some of the basic properties of semigroups that will later be used in the study of harmonic measures are the following.

Proposition 4.3. *Let D_t be a contraction semigroup of operators on a Banach space E, and let A be its infinitesimal generator. Then the domain $\mathscr{D}_A$ of A is a dense subspace of E. Moreover, D_t leaves $\mathscr{D}_A$ invariant, and if $f\in\mathscr{D}_A$, then*

$$\frac{d}{dt}D_t f = AD_t f = D_t\, Af,$$

and

$$D_t f - f = \int_0^t D_s Af \cdot dt.$$

The differentiation and integration operations in this statement are of functions from $[0,\infty)$ into the Banach space E. The usual rules of calculus also apply to these operations.

Proposition 4.4. *The infinitesimal generator of a contraction semigroup on E is a closed operator.*

A linear operator A defined in a linear subspace $\mathscr{D}_A$ of E is said to be closed if whenever $f_n\in\mathscr{D}_A$ is a sequence such that $f_n\to f$ and $Af_n\to Ag$, then $f\in\mathscr{D}_A$ and $Af = g$. Equivalently, A is closed if its graph is closed.

Proposition 4.5. *Let D_t be a contraction semigroup on the Banach space E, and let A be its infinitesimal generator. For every $g\in L$, the equation*

$$\lambda f - Af = g$$

admits exactly one solution $f \in \mathscr{D}_A$. This solution is given by the formula

$$f = R_\lambda g = \int_0^\infty e^{-\lambda t} D_t g \cdot dt.$$

The following theorem, due independently to Hille [19] and to Yosida [38], is the cornerstone of the theory of contraction semigroups.

Theorem 4.6. *Let A be a linear operator on a Banach space E with domain $\mathscr{D}_A$. Then A is the infinitesimal generator of a contraction semigroup of operators of E if and only if*:

(1) *the domain $\mathscr{D}_A$ is dense in E*;
(2) *A is a closed operator*; *and*
(3) *for all $\lambda > 0$, the operator $\lambda I - A$ is a bijection of $\mathscr{D}_A$ onto E, and has norm $||\lambda I - A|| \geqslant \lambda$.*

The construction of the diffusion semigroup associated to a Laplace operator Δ on a foliated space M will be based on Theorem 4.6. It is noted that Yosida already constructed the diffusion operator associated to the Laplacian in a compact Riemannian manifold, using some version of his theorem. The troublesome part is that in item (3) of Theorem 4.6, which may be seen as a cohomological condition: if it fails, on a compact manifold, then, by L^2-theory considerations, there is a smooth function h such that $\Delta^* h = \lambda h$, from what a contradiction is deduced. Item (2) is delicate because the operator Δ is not necessarily closed.

The tools from L^2-theory are not available in a general foliated space, thus such argument has no chance of being extended to this situation. In any case, it will next be shown that if M is a compact foliated space, of class at least C^3, and Δ is a Laplace operator, then there is a semigroup of operators D_t whose infinitesimal generator is an extension of Δ. The proof consists in showing that the operator Δ admits an extension A which satisfies the hypothesis of the Hille–Yosida theorem.

The first item of Theorem 4.6 is obviously satisfied, because the operator Δ is already defined on the space of continuous functions f on M which are of class C^2 on each leaf, and such that Δf is continuous on M.

The next proposition shows that the operator Δ almost satisfies the requirements of the third item of the list in Theorem 4.6.

Proposition 4.7. *Let M be a compact foliated space with a Laplace operator Δ, and let $\lambda > 0$. Then*

(1) *The norm $||\lambda I - \Delta|| \geqslant \lambda$.*
(2) *The operator $\lambda I - \Delta$ has dense image in $C(M)$.*

If f is a continuous function on M, of class C^2 along the leaves, and with Δf continuous, and if $\lambda > 0$, then, by a straightforward application of Lemma 3.4, it obtains that

$$\min_{x\in M}\,[\lambda f(x)-\Delta f(x)]\leqslant\lambda f(z)\leqslant\max_{x\in M}[\lambda f(x)-\Delta f(x)],$$

for all $z\in M$. This immediately implies that the operator $\lambda I-\Delta$ is injective and that its norm is at least λ. The inverse operator $(\lambda I-\Delta)^{-1}$ can therefore be defined on the range of $\lambda I-\Delta$. The calculation just made says that its norm $||(\lambda I-\Delta)^{-1}||\leqslant 1/\lambda$, and so it can be extended to a continuous linear operator on the closure of its domain. Therefore, the next task is to show that the domain of $(\lambda I-\Delta)^{-1}$ is dense in $C(M)$.

Lemma 4.8. *Let g be a Hölder continuous function on $\bar{D}\times Z$, and let $\lambda>0$. Then there exists a function f on $D\times Z$, of class C^2 on each plaque, with $f=\varphi$ on $\partial D\times Z$ and such that $\lambda f-\Delta f=g$.*

Proof. Define f to be, on each plaque $D\times\{z\}$, the solution to the problem $(D\times\{z\},g(\cdot,z),\varphi,\lambda)$. This function f will be continuous on $\bar{D}\times Z$ if the correspondence

$$z\in Z\mapsto f|D\times\{z\}\in C^{r,\alpha}(\bar{D})$$

is continuous. Since there are bounds on the Hölder norms of g, f, φ and the coefficients of the operators on $\bar{D}\times Z$ in an appropriate space $C^{r,\alpha}(\bar{D})$, the family of functions $\{f_z\mid z\in Z\}$ is precompact in $C^{r,\alpha}(\bar{D})$, for some α. On the other hand, the family of operators Δ_z is compact, as is the continuous image of the compact space Z in a suitable Banach space. Therefore, any limit of the family $\{f_z\}$ must be again a member of this family. That is, the induced map $Z\to C^{r,\alpha}(\bar{D})$ is continuous. $\square$

Continuing with the assumption that M is compact and $\lambda>0$, let g be a Hölder continuous function on M, and let κ be a constant such that $g\geqslant\lambda\kappa$ on M. For each leaf L of M, let f_L be the function on L solving the equation $\lambda f_L-\Delta f_L=g$, which was constructed as

$$f_L=\sup_D\{f_D\mid D\text{ bounded regular domain in }L\},$$

and where f_D is the solution to $\lambda f-\Delta f=g$ on D with $f=\kappa$ on ∂D. Let f_i be the function on M which agrees with f_L on each L. This function is bounded on M, of class C^2 on each leaf, and satisfies the equation $\lambda f_i-\Delta f_i=g$ on M.

Proposition 4.9. *The function f_i is lower semicontinuous on M.*

Proof. Let f denote f_i. It needs to be shown that $\liminf_{y\to x}f(y)\geqslant f(x)$, for every x in M. Let L be the leaf containing x and let $L'\to M$ be the holonomy cover of L, embedded in M via L. There is a vector bundle E over L' and a lamination N in a neighborhood of the zero section of E which is transverse to the fibers and has L' as one leaf. Since L' has no holonomy as a leaf of N, given a relatively compact domain D in L', there is a transversal Z through x and an embedding of the trivial foliation $D \times Z$ into N. Let f and g denote the functions lifted to E. Since the map $E \to M$ is locally a diffeomorphism on the leaves, the operator Δ lifts to the foliated space E, due to its local nature (Lemma 1.3).

On $\bar{D} \times Z$ there exists, by Lemma 4.8, a continuous function h which satisfies $\lambda h - \Delta h = g$ and equals the constant κ on $\partial D \times Z$. Since $f_D \leqslant f$, also $\kappa \leqslant f$ on $\partial D \times Z$, and the comparison Lemma 1.8 implies that $h \leqslant f$ on $D \times Z$. Since the function h is continuous

$$h(x) = \lim_{y \to x} h(y) \leqslant \liminf_{y \to x} f(y).$$

Finally, by the construction of f as $\sup_D f_D$, h equals f_D on D, and increases to f as D increases to L'. Therefore, given $\varepsilon > 0$ there is a domain D in L' containing x such that the corresponding function h satisfies $0 \leqslant f(x) - h(x) \leqslant \varepsilon$ and so $f(x) \leqslant \liminf_{y \to x} f(y) + \varepsilon$, from what the result follows. $\square$

Suppose that $g \leqslant \lambda\kappa$ on M, where κ is a fixed constant (perhaps different from the above). A solution f_L to the equation $\lambda f - \Delta f = g$ on each leaf can be constructed as

$$f_L = \inf_D \{f'_D \mid D \text{ bounded regular domain in } L\},$$

where f'_D solves the corresponding problem in the regular domain $D \subset L$ with boundary conditions κ on ∂D. The function f_u on M which agrees with each of these f_L on the leaf L is also a solution to the same equation on M. An argument dual to that in Proposition 4.9 shows that f_u is upper semicontinuous on M. Moreover, $f_u \geqslant f_i$. Indeed, if there exists a point x in M such that $f_u(x) + \varepsilon < f_i(x) - \varepsilon$, then, by the construction of these functions, there exists a domain D containing x such that $f'_D(x) < f_D(x)$, contradicting Lemma 1.8.

Therefore the difference $h = f_u - f_i$ is bounded, non-negative, upper semicontinuous on M, of class C^2 on each leaf, and satisfies the equation $\Delta h = \lambda h$. A bounded upper semicontinuous function on a compact space attains an absolute maximum. If x denotes a point where h reaches its maximum, then $\Delta h(x) \leqslant 0$, implying $h(x) \leqslant 0$, hence that $h \equiv 0$ on M because $h \geqslant 0$. Therefore the functions $f_u = f_i$, and they are continuous.

Proposition 4.10. *Let g be a Hölder continuous function on M. Then there exists one and only one continuous function f on M, of class C^2 on each leaf, which satisfies*

$$\lambda f - \Delta f = g$$

on M.

The identity $\Delta f = \lambda f - g$ implies that Δf is also continuous. It easily follows from considerations on flow boxes and approximation arguments like those in the proof of

Proposition 4.9 that if g is locally Hölder, then the solution f is of class C^2 on the foliated space.

The second item in the statement of Theorem 4.6 will now be discussed, and the infinitesimal generator of Δ and its domain will be identified.

Lemma 4.11. *Let M be a foliated space, not necessarily compact, and let Δ be a Laplace operator on M. If f_n is a sequence of bounded functions such that $f_n \to 0$ and $\Delta f_n \to g$, both limits being uniform, then $g = 0$.*

Proof. Suppose that $f_n \to 0$ and $\Delta f_n \to g$, both uniformly. If g fails to be identically zero, then there is a closed disc D contained in a leaf such that $g > 0$ on D (after changing signs, if necessary). Let h be a smooth, compactly supported function on D, which is non-negative on D and positive on a smaller open subset of D. A contradiction is then reached, for on the one hand

$$\int_D h\Delta f_n \to \int_D hg > 0,$$

and, on the other hand, by Green's formula of Proposition 3.1 (Δ^* being the formal adjoint of Δ)

$$\int_D h\Delta f_n = \int_D f_n \Delta^* h \to 0. \qquad \square$$

This lemma says that the operator Δ is closable, hence that it admits closed extensions. The right one will be now chosen.

Define an operator A on a subspace of $C(M)$ as follows. Let f, g be continuous functions on M. Then $Af = g$ if the following identity holds true:

$$\int_D g\varphi = \int_D f \Delta^* \varphi,$$

for every regular domain D in a leaf of M and every function $\varphi \in D^2_c(D)$.

Let $\mathscr{D}_A$ denote the collection of functions $f \in C(M)$ for which Af can be defined as above. That is, $\mathscr{D}_A$ consists of all continuous functions f on M for which Δf exits on each leaf in the distribution sense, and can be represented by a continuous function on M. It is evident that $\mathscr{D}_A$ is a linear subspace of $C(M)$ which contains all the continuous functions $f \in C(M)$ which are C^2 on each leaf, with Δf continuous.

Proposition 4.12. *The operator A is well defined, it is closed, and it extends Δ.*

Proof. Suppose that f is a continuous function on M and that there exist two continuous functions g_1 and g_2 which serve the definition of Af. Then

$$\int_D (g_1 - g_2)\varphi = 0,$$

for every domain D and every function $\varphi \in C_c^k(D)$. Hence $g_1 = g_2$ on D, by continuity, and thus also on M.

To show that A is closed, suppose that f_n is a sequence of functions which converges uniformly to a function f and such that Af_n converges uniformly to g. Since C^2 functions are dense in $\mathscr{D}_A$, it may be assumed that each f_n is C^2 and that $\Delta f_n \to g$. From the integral identity it follows that g serves the definition of Af, for if $\varphi \in C_c^k(D)$, then

$$\begin{aligned}\int_D g\varphi &= \lim_n \int_D (\Delta f_n)\varphi \\ &= \lim_n \int_D f_n \Delta^* \varphi \\ &= \int_D f \Delta^* \varphi.\end{aligned}$$

Finally, it is clear that A agrees with Δ on the space of continuous functions f on M which are C^2 on the leaves and with Δf continuous. □

Proposition 4.13. *For each* $\lambda > 0$, *the operator* $\lambda I - A$ *has image equal to* $C(M)$, *it is injective, and has norm* $||\lambda I - A|| \geqslant \lambda$.

Proof. Let g be a continuous function on M, and let $\{g_n\}$ be a sequence of locally Hölder continuous functions on M which converges uniformly to g. For each g_n there exists a unique function f_n such that $\lambda f_n - \Delta f_n = g_n$.

The identity $\lambda(f_n - f_m) - \Delta(f_n - f_m) = g_n - g_m$ implies that

$$\lambda||f_n - f_m|| \leqslant ||g_n - g_m||.$$

Hence, since $\{g_n\}$ is a Cauchy sequence in $C(M)$, the sequence $\{f_n\}$ is also Cauchy on $C(M)$, and so it converges uniformly to a continuous function f.

It needs to be shown that $Af = g$ according to the definition of A. Let D be a domain on a leaf and let $\varphi \in C_c^k(D)$. Then

$$\int_D g_n \varphi = \int_D f_n \Delta^* \varphi$$

for each n, and the bounded convergence theorem implies that

$$\int_D g\varphi = \int_D f \Delta^* \varphi.$$

Finally, if $(\lambda - A)f = 0$, then it follows from regularity theory of weak solutions to elliptic equations that f is at least of class C^2 along the leaves, and so $f \equiv 0$ by Lemma 3.4, proving injectivity. □

The Hille–Yosida theorem (Theorem 4.6) applies to the operator A, therefore:

Theorem 4.14. *Let M be a compact foliated space and let Δ be a Laplace operator on M. Then there is a semigroup D_t on $C(M)$ whose infinitesimal generator is the extension on Δ considered in Proposition* 4.12.

To finish this section, the differences of this approach with that presented by Garnett [14] must be pointed out. First of all, she considers only the Laplace operator of a metric tensor. On a Riemannian manifold L of bounded geometry, such operator has associated to it the so-called heat kernel $p(x,y;t)$ which gives rise to a diffusion operator D_t defined on bounded functions on L by

$$D_t f(x) = \int_L f(y)p(x,y;t) \cdot \mathrm{vol}(y).$$

In fact, the diffusion semigroup D_t (of the manifold L) is completely characterized by such formula on the space $C_0(L)$ of continuous functions on L vanishing at infinity. Garnett attacks the problem of continuity of D_t in a different way. When the foliated space M is compact and is endowed with a metric tensor, every leaf becomes a manifold of bounded geometry. If f is a continuous function on M, then the restriction f_L of f to L is a bounded continuous function, so it makes sense to define $D_t f$ to be the aggregate of the diffusions of the functions f_L on the corresponding leaf L. Then she provides an argument to show that the aggregate function $D_t f$ is continuous on M if f is continuous.

The construction of D_t just carried out here is indirect, and it should be argued that the two constructions yield the same result. In fact, it will be shown that the method of proof presented provides an independent construction of the heat kernel of each leaf. It is not obvious in principle, because not every function on a leaf which vanishes at infinity is the restriction of a continuous function on M (that would be the case if L was a proper leaf), much less a bounded function on L is a restriction.

Let M be a foliated space and let L be a leaf of M. Let L' be a manifold isomorphic to L. Each object $\mathcal{O}$ (point, subset, function, etc.) associated to L defines a corresponding object $\mathcal{O}'$ associated to L'. Let M' denote the disjoint union of M and L', endowed with the topology whose system of neighborhoods is as follows. A neighborhood of a point of L' is a manifold neighborhood; a neighborhood of a point x of M is of the form $U \cup (U \cap L)'$, where $U \subset M$ is a neighborhood of x in M. It is evident that M' is a foliated space of the same smoothness as M and having the manifold L' as a proper leaf.

The space M' contains a compactification of L', namely that whose boundary is the closure of L in M. (If L is compact, then M' is the disjoint union of M and L'; this is a degenerate situation which may safely be removed from the present discussion.) This compactification of L' (or of L) can be described by an algebra of bounded continuous functions on L which contains the algebra of functions which vanish at infinity. Every function f on L which vanishes at ∞ has a unique extension to a function on M', namely the extension is f on L' and as 0 on M. Analogously, a continuous function f on M admits a canonical extension f' to M' obtained by defining $f'(x')$ to be $f(x)$ where $x' \in L'$ corresponds to $x \in L$.

Proposition 4.15. *Let M be a compact foliated space, and let L be a leaf of M. The Banach space $C(M')$ contains isometric copies of the Banach spaces $C(M)$ and $C_0(L')$.*

A Laplacian operator Δ on M has a canonical extension Δ' to M', which is obtained by the local expression for Δ. The preceding discussion, applied to the extended operator Δ' on the foliated space M', produces a diffusion semigroup D'_t on the Banach space $C(M')$ which extends the semigroup on the Banach space $C(M)$ defined by Δ. Since the operator Δ' leaves the subspaces $C(M)$ and $C_0(L')$ invariant, the diffusion semigroup associated to Δ' preserves the subspaces $C_0(L')$ of $C(M')$, and it obviously agrees with the old D_t on $C(M)$.

If M is a foliated space and f is a continuous bounded function on M, then $D_t f$ is the solution to the differential equation

$$\frac{d}{dt} D_t f = \Delta D_t f$$

in $C(M)$ with initial conditions $D_0 f = f$. If L is a leaf of M, then the theory of elliptic differential equations on a manifold say that such function $D_t f$ on L is given by a smooth kernel $p(x, y; t)$:

$$D_t f(x) = \int_L f(y) p(x, y; t) \cdot \mathrm{vol}(y)$$

where $\mathrm{vol}(y)$ is the volume density of the metric tensor associated to the Laplace operator Δ on L. Therefore the two constructions lead to the same result because they agree on compactly supported functions on each leaf.

Proposition 4.16. *Let M be a compact foliated space and let Δ be a Laplace operator on M. Then the diffusion semigroup D_t associated to Δ acts on a continuous function f on M by*

$$D_t f(x) = \int_L f(y) p(x, y; t) \cdot \mathrm{vol}(y),$$

where L is the leaf through x and $p(x, y; t)$ is the fundamental solution (heat kernel) of the heat equation $\frac{d}{dt} - \Delta$ on L.

The semigroup D_t defines a family of measures P_x on M via the positive linear functional

$$f \in C(M) \mapsto D_t f(x).$$

By the previous discussion, this measure P_x is the image of the measure $p(x, y; t)\,\mathrm{vol}(y)$ in L under the inclusion $L \to M$. Since $D_t f(x)$ can be represented by a probability measure on M, the semigroup D_t has a canonical extension to a semigroup on the Banach space of bounded Borel measurable functions with the supremum norm. It then makes sense to apply D_t to the characteristic function χ_B of a Borel subset B of M.

Standing Hypothesis. From now on, the foliated space and the Laplace operator under consideration will satisfy the smoothness conditions so that the results of this section apply.

5. Characterization of harmonic measures

Two other characterizations of harmonic measures will be described in this section. One of them will show the formal similarity they have with transverse invariant measures. If the foliated space has a metric tensor along the leaves, a transverse invariant measure combines with the Riemannian measure on the leaves to give a measure on the foliated space. Such measure on the foliated space is called totally invariant, and it will be shown that totally invariant measures can be described in terms of the diffusion semigroup associated to the Laplace operator of the metric tensor.

Let D_t be the diffusion semigroup associated to the Laplacian Δ on M. The infinitesimal generator of D_t will also be denoted by Δ. The action of D_t on the Banach space $C(M)$ induces, by duality, an action on the space of measures on M, which will also be denoted by D_t. That is, if m is a measure on M, then $D_t m$ is the measure defined by

$$\int_M f(x) \cdot (D_t m)(x) = \int_M D_t f(x) \cdot m(x).$$

This action preserves probability measures on M because D_t is a positive operator and $D_t 1 = 1$. The following results says that harmonic measures are fixed points of this action. This is of course contained in [14]; the proofs using diffusion semigroups are much easier.

Proposition 5.1. *A measure m on M is harmonic if and only if $D_t m = m$.*

Proof. Suppose that m is D_t invariant. If f is a smooth function then Δf is the uniform limit of the family $(D_t f - f)/t$ as $t \to 0$. Therefore, $\int_M \Delta f \cdot m = 0$ by the dominated convergence theorem.

Conversely, suppose that m is harmonic. If f is a smooth function on M, then the function

$$t \in [0, \infty) \mapsto D_t f \in C(M)$$

is continuous for $t \geqslant 0$ and differentiable for $t > 0$, and these two properties are inherited by the function

$$t \mapsto \int_M D_t f \cdot m,$$

because integration is a continuous linear functional. But, by Proposition 4.3,

$$\frac{d}{dt} D_t f = \Delta D_t f,$$

which causes

$$\frac{d}{dt}\left(t \mapsto \int_M D_t f \cdot m\right) = \int_M \Delta D_t f \cdot m = 0.$$

Therefore, $t \mapsto \int_M D_t f \cdot m$ is a constant function; by continuity, it must be equal to

$$\int_M D_0 f(x) \cdot m(x) = \int_M f(x) \cdot m(x). \qquad \square$$

The second characterization of harmonic measures is local and shows the similarity that they have with transverse invariant measures. In fact, a corollary is that a transverse measure, when combined with the Riemann volume density along the leaves, is a harmonic measure.

Proposition 5.2. *A measure m on M is harmonic if and only if on any given flow box $D \times Z$ it admits a decomposition of the following form: if f is a function with support in $D \times Z$,*

$$\int_M f(y) \cdot m(y) = \int_Z \left(\int_{D \times \{z\}} f(x,z) h(x,z) \cdot \mathrm{vol}_z(x) \right) \cdot v(z),$$

where v is a measure on the transversal Z, $h(\bullet, z)$ is Δ^*-harmonic on $D \times \{z\}$ for v-almost all $z \in Z$, Δ^* is the adjoint operator to Δ, and $\mathrm{vol}_z(x)$ *is the volume density of the plaque $D \times \{z\}$.*

The local decomposition is provided by the disintegration of the measure with respect to the fibration $D \times Z \to Z$, which is constant on the leaves. This allows to find a measure v on Z and a measurable assignment of a probability measure λ_z on D to v-almost all z in Z. (This is a point where the local compactness of M is used.) The rest is a direct consequence of the regularity results for weak solutions to elliptic differential equations, which permit to conclude that for v-almost all z in Z, the measure λ_z can be represented as $\lambda_z = h(\bullet, z)\, \mathrm{vol}_z(\cdot)$, where $h(\bullet, z)$ is a Δ^*-harmonic function on $D \times \{z\}$.

This local description says that the relation between the measure class of a harmonic measure and Riemannian leaf measures is that a property that holds almost everywhere with respect to a harmonic measure will hold leafwise almost everywhere with respect to Riemannian volume for almost all leaves.

It is also convenient to observe that if m is a measure on M (not necessarily harmonic) then the measures $D_t m$ and $D_s m$ are in the same measure class (i.e., are absolutely continuous with respect to each other) for all $s, t > 0$. This is so because the heat kernel $p(x, y; t)$ $(t > 0)$ associated to the Laplace operator Δ is a positive smooth density on each leaf, and so $D_t m(B) = 0$ if and only if $m(\{x \mid \int_{L_x \cap B} p(x, y; t) \cdot \mathrm{vol}(y) > 0\}) = 0$. This says that for each leaf L, except for a set of leaves of m-measure zero, the Riemann measure of $L \cap B$ in L is zero. The following summarizes all that will be required from the discussion in [14].

Corollary 5.3. *A harmonic measure m on M is smooth leafwise, that is, a Borel subset B of M is m-null if and only if $B \cap L$ is a null set in L, for all leaves L except a m-null set of leaves.*

The diffusion operator D_t converts measures into smooth measures, and it preserves the measure class of a measure m if and only if m is smooth.

The local decomposition of a harmonic measure m is not unique. In another flow box $D' \times Z'$, this decomposition of m would be of the form

$$m \equiv \int_{Z'} \left(\int_{D' \times \{z'\}} h'(x', z') \cdot \mathrm{vol}(x') \right) \cdot v'(z').$$

If this box intersects the previous one $D \times Z$, then there is a partially defined holonomy homeomorphism $z : z' \in Z' \mapsto z(z') \in Z$, and the measures $z_* v' = v' \circ z^{-1}$ and v are in the same measure class. Indeed, if the coordinate change from $U \times Z$ to $U' \times Z'$ is written as

$$(x', z') = (x'(x, z), z'(z))$$

then, after testing against functions with support in the intersection of the two charts, it is seen that the local expressions for the measures must satisfy

$$h'(x'(x, z), z'(z)) \cdot \mathrm{vol}(x) v'(z'(z)) = h(x, z) \cdot \mathrm{vol}(x) v(z).$$

It follows from the Radon–Nikodym theorem that

$$v'(z'(z)) = \frac{dv'(z'(z))}{dv(z)} \cdot v(z)$$

and $v'(z'(z)) = ((z')^{-1})_* v'(z) = z_* v'(z)$, where z' is the transformation $z \in Z \mapsto z'(z) \in Z'$ inverse to $z : z' \mapsto z(z')$. Thus, on the domain of the holonomy transformation $z : Z' \to Z$, the following identity holds true:

$$h(x, z) = h'(x'(x, z), z'(z)) \frac{dz_* v'}{dv}(z).$$

The ratio h/h' is constant along almost all plaques. Two local decompositions of m provide measures on the same class, so that the ratio $h/h' > 0$ almost everywhere, and this means that the corresponding harmonic functions h and h' are related by

$$d \log h = d \log h'$$

on ν-almost every plaque $D \times \{z\}$, where d is the differential along the leaf coordinates. Thus $d \log h$ is the local expression of a measurable one-form η along the leaves of M, called the *modular form* of the measure m. This modular form measures the transverse volume distortion. Indeed, if σ is a path in a leaf going from a point x to a point y, then $\exp \int_\sigma \eta$ is the distortion of the measure done by the holonomy transformation along σ in going from x to y.

Corollary 5.4. *The modular form η satisfies*

$$\int_M \operatorname{div} Y(x) \cdot m(x) = -\int_M \eta(Y)(x) \cdot m(x)$$

for every vector field Y along the leaves with integrable divergence (div *with respect to the metric tensor associated to the Laplacian* Δ).

Corollary 5.5. *The harmonic measure m is totally invariant if and only if its modular form $\eta = 0$.*

This information can be conveniently summarized in the following way. Let $\mathcal{G}$ denote the holonomy groupoid of M. As a set, points of $\mathcal{G}$ are equivalence classes of paths in leaves, two paths c and c' being equivalent if they have the same endpoints and the loop $c'c^{-1}$ has trivial holonomy. There is a projection $r : \mathcal{G} \to M$ sending a path c in $\mathcal{G}$ to its initial point $c(0)$. If $x \in M$, then $\mathcal{G}_x = r^{-1}(x)$ is the holonomy cover of the leaf through x.

Let $\mathcal{H}_+$ be the collection of pairs (h, c), where $c \in \mathcal{G}$ and h is a positive harmonic function on $\mathcal{G}_{r(c)}$. There is a projection map $\mathcal{H}_+ \to M$ sending (h, c) to $r(c)$, and an evaluation map $\mathcal{H}_+ \to \mathbb{R}_+$ sending (h, c) to $h(c)$, as $c \in \mathcal{G}_{r(c)}$. There is a natural action of the positive real numbers on $\mathcal{H}_+$, namely, t acts by sending (h, c) to (th, c). There is also a natural equivalence relation $\sim$ induced by the action of the groupoid $\mathcal{G}$ on itself, which is obviously compatible with the action of $\mathbb{R}^+$.

Let $\mathcal{M}$ be the quotient space of $\mathcal{H}_+$ by the $\mathbb{R}^+$-action. Then the equivalence classes of the relation induced by $\sim$ on $\mathcal{M}$ are the leaves of a foliated space structure on $\mathcal{M}$. To make this precise, a topology must be introduced on $\mathcal{M}$. The end result is a fibration $\pi : \mathcal{M} \to M$, with the fiber over the point x being a copy of the Martin space (rays in the cone of positive harmonic functions) of the holonomy cover of the leaf through x. The Martin space has a canonical point, that corresponding to the positive constants.

If m is a harmonic measure on M, then there is a map $j : M \to \mathcal{H}$, defined almost everywhere on M, and which sends the point x to the pair (h, c_x), where h is a representative of the Jacobian of m in the leaf through x, and c_x is the constant path at x. The reason that this is well defined, on a conull subset of M, is that the Radon–Nikodym derivative of the identity is equal to 1 almost everywhere.

6. The heat equation, Brownian motion and stochastic processes

Let M be a foliated space with Laplace operator Δ and associated diffusion semigroup D_t. Associated to these objects, there is a space $\Omega(M)$ with a continuous semi-flow of transformations θ, and this section is devoted to their description. The space $\Omega(M)$ is fibered over M, and the operator Δ will be used to define a family of probability measures P_x on the fibers. A harmonic measure m on M induces an invariant measure for this flow, by integration of the measures P_x over m.

The σ-algebra of Borel subsets of M will be denoted by $\mathfrak{M}$. As M is metrizable, the space of continuous maps from the infinite half-line $[0, \infty)$ into M, endowed with the compact-open topology, is also metrizable. Let $\Omega(M)$ denote the subspace consisting of those continuous maps $[0, \infty) \to M$ with image fully contained in a single leaf. It is a closed subspace, hence metrizable, complete and separable. Let the map $\pi_t : \Omega(M) \to M$ denote the position at time t, $\pi_t(\omega) = \omega(t)$.

The topology of $\Omega(M)$ is separable and metrizable. If d is a (bounded) distance on M, then the expression

$$d(\omega, \omega') = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{0 \leqslant t \leqslant n} d(\omega(t), \omega'(t))$$

defines a bounded metric on $\Omega(M)$ such that the evaluations maps $\omega \mapsto \omega(t)$ are Lipschitz.

The construction of the measures on $\Omega(M)$ needs to be done first in the space of all maps from the half-line $[0, \infty)$ into M, which is denoted by $M^{[0,\infty)}$. The natural topology of this space is the product topology, but its associated Borel σ-algebra is too large for most purposes. Instead, the σ-algebra $\mathfrak{C}$ generated by the cylinder sets needs to be considered, these being the sets of the form

$$C = \{\omega \in M^{[0,\infty)} \mid \omega(t_1) \in B_1, \dots, \omega(t_n) \in B_n\},$$

where $B_1, \dots, B_n$ are Borel subsets of M, and $0 \leqslant t_1 < \cdots < t_n$ is a finite set of times. That is, C consists of all elements of $M^{[0,\infty)}$ which can be found within B_i at time t_i.

The structure of the measure space $(M^{[0,\infty)}, \mathfrak{C})$ is best understood by viewing it as an inverse limit. To do so, let the collection of finite subsets of $[0, \infty)$ be partially ordered by inclusion. Associated to each finite subset F of $[0, \infty)$ is the measure space $(M^F, \mathfrak{M}^F)$, where $\mathfrak{M}^F$ is the Borel algebra of the product topology on M^F. Each inclusion of finite sets $E \subset F$ canonically defines a projection $\pi_{EF} : M^F \to M^E$ which drops the finitely many coordinates in $F \backslash E$. These projections are continuous, hence measurable, and consistent, for if $E \subset F \subset G$, then $\pi_{EF} \circ \pi_{FG} = \pi_{EG}$. The family $\{M^F, \pi_{EF} \mid E, F \subset [0, \infty) \text{ finite}\}$ is an inverse system of spaces, and its inverse limit is

$M^{[0,\infty)}$ with canonical projections $\pi_F : M^{[0,\infty)} \to M^F$. The σ-algebra generated by the cylinders sets is the smallest one making all the projections π_F measurable.

For each $x \in M$, a probability measure P_x on the measure space $(M^{[0,\infty)}, \mathfrak{M})$ will now be defined. If $F = \{0 \leqslant t_1 < \cdots < t_n\}$ is a finite subset of $[0, \infty)$ and $C^F = B_1 \times \cdots \times B_n$ is a cylinder set of $(M^F, \mathfrak{M}^F)$, define

$$P_x^F(C^F) = D_{t_1}(\chi_{B_1} D_{t_2}(\chi_{B_2} \cdots \chi_{B_{n-1}} D_{t_n - t_{n-1}} \chi_{B_n}))(x),$$

where χ_{B_i} is the characteristic function of B_i and D_t is the diffusion operator associated to the Laplace operator Δ on M.

It is an obvious consequence of the semigroup property of D_t that if $E \subset F$ are finite subsets of $[0, \infty)$ and C^E is a cylinder subset of M^E, then

$$P_x^E(C^E) = P_x^F(\pi_{EF}^{-1}(C^E)).$$

It then follows that a probability measure P_x on $(M^{[0,\infty)}, \mathfrak{C})$ can be defined so that it is consistent with the inverse limit structure. This measure P_x gives the set of paths ω with $\omega(0) = x$ total probability.

Cylinder sets can be used to define σ-algebras on $\Omega(M)$ via the inclusion into $M^{[0,\infty)}$. Even when the topologies are unrelated, it happens that the Borel algebra of $\Omega(M)$ as a polish space is the one generated by the sets $C \cap \Omega(M)$, when C runs over all cylinder sets in $M^{[0,\infty)}$. Therefore, every probability measure on $(\Omega(M), \mathfrak{B})$ is uniquely determined by its values on the cylinder sets.

Going from a probability measure on $M^{[0,\infty)}$ to one on $\Omega(M)$ is a more difficult step. It requires to show that the outer measure of $\Omega(M)$ is one with respect to each probability measure P_x. In general, the fact that the semigroup D_t preserves continuous functions only guarantees that the paths of the process are right continuous.

To show that these probability measures P_x give full measure to the smaller space of continuous paths $\Omega(M)$ requires further analytical work. For an arbitrary operator on a foliated space M this would involve two steps: a first one for the passage from $M^{[0,\infty)}$ to the space of continuous paths in M, and a second one for going to the smaller class of continuous leaf paths $\Omega(M)$. The first one can be taken under a condition on the measures P_x of the following form:

$$\lim_{t \to 0} \frac{1}{t} \sup_{x \in M} P_x[\omega(t) \notin B_M(x, \varepsilon)] = 0,$$

for each $\varepsilon > 0$. A condition of the form

$$\lim_{t \to 0} \frac{1}{t} \sup_{x \in M} P_x[\omega(t) \notin B_{L_x}(x, \varepsilon)] = 0, \tag{$*$}$$

for each $\varepsilon > 0$, guarantees the second restriction on paths. A proof of the continuity of sample paths under $(*)$ can be constructed using those techniques that work in the case of euclidean space. A fuller discussion of these matters, as well as background material, is to be found in [12,13,23,35] for instance.

If Δ is a Laplace operator on a compact foliated space, then the normal estimate of Cheng-Li-Yau [7], quoted in Theorem 1.14, guarantees such condition. Indeed, as M is compact, there are global bounds for the curvature an injectivity radius of the leaves, and for the potential of the symmetrized operator, so the estimate of Theorem 1.14 holds uniformly for all the leaves. Moreover, it is always possible to construct a distance function d_M on M which respect to which all the leaves inclusions a Lipschitz maps of distortion $\leqslant 1$; thus the second condition implies the first. Once the normal estimate for the heat kernel is available, a direct calculation shows that the second condition $(*)$ holds. This calculation will be carried out in Section 8, because it will be needed for other applications.

The construction of the Markov process associated to an elliptic operator on a compact foliated space requires the introduction of two families of σ-algebras. Let $\mathfrak{B}_t$, $t \geqslant 0$, be the σ-algebra which is generated by all cylinder sets in $\Omega(M)$ whose associated sequence $t_1, \ldots, t_n$ is bounded above by t. Hence $\mathfrak{B}_t$ keeps track of happenings up to time t. It is clear that $\mathfrak{B}_s \subset \mathfrak{B}_t$ if $s \leqslant t$. and that the projection π_t is measurable with respect to $\mathfrak{B}_t$ for each $t \geqslant 0$. The σ-algebra $\mathfrak{B}_{t+}$ defined as the intersection of all $\mathfrak{B}_s$ with $s > t$, i.e.,

$$\mathfrak{B}_{t+} = \bigcap_{s>t} \mathfrak{B}_s,$$

keeps track of events that happen immediately after t.

The collection $(\Omega(M), \mathfrak{B}_t, \pi_t)$ is called an stochastic process with values on $(M, \mathfrak{M})$. The family of measures $\{P_x \mid x \in M\}$ is such that

(1) P_x is a probability measure on $\Omega(M)$ concentrated on the fiber $\pi_0^{-1}(x)$;
(2) For each $B \in \mathfrak{B}$, the map $x \mapsto P_x(B)$ is measurable; and
(3) for every $x \in M$, $s \geqslant t \geqslant 0$, Borel sets $A \in \mathfrak{B}$ and $B \subset M$,

$$P_x[A \cap \{\omega \mid \omega(s) \in B\}] = \int_A P_{v(t)}[\omega \mid \omega(s-t) \in B] \cdot P_x(v).$$

All this constitutes the Markov process associated to the Laplace operator Δ on M. Item (3) above is called the Markov property, and is a reflection of the semigroup property of the diffusion semigroup operators D_t. There is a stronger version of it which goes by the name of strong Markov property and which will be mentioned after introducing the required terminology.

The space $\Omega(M)$ supports a dynamical system which is closely connected to the foliation dynamical system of M. Let $\theta = \{\theta_t \mid t \geqslant 0\}$ denote the semigroup of shift transformations of $\Omega(M)$ defined by $\theta_t(\omega)(s) = \omega(s+t)$. It satisfies $\pi_0 \circ \theta_t = \pi_t$. This semigroup of operators permits to study recurrence properties of the foliation dynamical system M as if it was given by a one dimensional flow.

A stopping time for the Brownian motion on $\Omega(M)$ is a function

$$\tau : \Omega(M) \to [0, \infty]$$

which is adapted to the filtration $\mathfrak{B}_t$, that is, for each $t>0$, the set of paths

$$\{w \mid \tau(\omega) < t\}$$

is $\mathfrak{B}_t$-measurable. Equivalently, $\{\tau(\omega) \leqslant t\}$ is $\mathfrak{B}_{t+}$-measurable for all $t \geqslant 0$.

The simplest example of stopping time is the constant time $\tau(\omega) = t$ for all ω. More important examples are the first entrance and first exit times from subsets of M. The hitting time τ_B of a subset B of M is defined as

$$\tau_B(\omega) = \inf\{t > 0 \mid \omega(t) \in B\}$$

with the convention that the infimum of the empty set is ∞. It is easily verified that hitting times of F_σ-subsets of M are stopping times. The exit time T_B of a subset B of M is defined as the hitting time of its complement, $T_B = \tau_{M \setminus B}$.

Corresponding to a stopping time τ is the σ-field $\mathfrak{B}_{\tau+}$ defined to be the collection of events A such that $A \cap \{\tau \leqslant t\} \in \mathfrak{B}_{t+}$ for all $t \geqslant 0$. It is easily verified that τ is $\mathfrak{B}_{\tau+}$-measurable.

The strong Markov property says the following. See [12], [33] for proofs.

Theorem 6.1. *Let F be a measurable function on $\Omega(M)$ and let τ be a stopping time. Then*

$$E_x[F \circ \theta_\tau \mid \mathfrak{B}_{\tau+}](\omega) = E_{\pi_\tau(\omega)}[F],$$

for P_x-almost all ω in $\{\tau < \infty\}$. More generally, if $F(t, \omega)$ is a bounded measurable function on $[0, \infty) \times \Omega(M)$, then

$$E_x[F(\tau, \theta_\tau) \mid \mathfrak{B}_{\tau+}](\omega) = E_{\pi_\tau(\omega)}[F(\tau(\omega), \bullet)],$$

for P_x-almost all ω in $\{\tau < \infty\}$.

The following result, known as Dynkin's formula, will be useful (see [13] for a proof). A generalization of this formula will be given in Section 8.

Theorem 6.2. *Let $S \leqslant T$ be stopping times with $P_\bullet[T < \infty] = 1$. If f is a function in the domain of the infinitesimal generator, then*

$$E_x[f \circ \pi_T] - E_x[f \circ \pi_S] = E_x\left[\int_S^T \Delta f(\omega(s)) \cdot ds\right].$$

Remark. In particular, by Proposition 4.16, this formula applies to functions f which belong to the space $C_0^2(L)$ of C^2 functions on a leaf L vanishing at ∞.

A measure m on M induces a measure μ on $\Omega(M)$, which is defined by the formula

$$\int_{\Omega(M)} F(\omega)\cdot\mu(\omega) = \int_M \left(\int_{\pi_0^{-1}(x)} F(\omega)\cdot P_x(\omega)\right)\cdot m(x).$$

In particular, if f is a continuous function on M, then, by the Markov property,

$$\begin{aligned}\int_{\Omega(M)} f\circ\pi_t(\omega)\cdot\mu(\omega) &= \int_M \left(\int_{\pi_0^{-1}(x)} f\circ\pi_t(\omega)\cdot P_x(\omega)\right)\cdot m(x)\\ &= \int_M D_t f(x)\cdot m(x).\end{aligned}$$

Together with the fact that a measure m on M is harmonic if it is diffusion invariant, the following is obtained.

Proposition 6.3. *If m is a diffusion invariant measure on M, then the measure μ on $\Omega(M)$ is invariant under the semigroup θ of shift transformations.*

Obviously, not every shift invariant measure on $\Omega(M)$ arises from a harmonic measure on M, but those that do can be intrinsically described as Gibbs measures.

To relate the ergodic properties of the two dynamical systems M and $\Omega(M)$, the introduction of the tail σ-algebra on $\Omega(M)$ is required.

Let $\mathfrak{B}^t$ denote the σ-algebra on $\Omega(M)$ generated by all the cylinder sets with time sequence bounded below by t, that is, sets of the form $B=\pi_F^{-1}(C)$, where $F\subset[0,\infty)$ is a finite set with $\min F\geqslant t$. The tail σ-algebra is the algebra $\mathfrak{B}^\infty$ on $\Omega(M)$ defined as

$$\mathfrak{B}^\infty = \bigcap_{t\geqslant 0}\mathfrak{B}^t.$$

The relevance of this σ-algebra is that it contains all the events which are invariant under θ_t; its properties are related to those of the invariant sets of M. Indeed, let $B\subset\Omega(M)$ be θ-invariant, i.e., $\theta_t^{-1}(B)=B$ for all $t\geqslant 0$. Since the map $\theta_t:(\Omega(M),\mathfrak{B}^s)\to(\Omega(M),\mathfrak{B}^{s+t})$ is measurable, it follows that $B\in\mathfrak{B}^\infty$. Hence,

Proposition 6.4. *The algebra $\mathfrak{B}^\infty$ contains the σ-algebra of θ-invariant Borel subsets of $\Omega(M)$. More generally, it contains all the subsets which are invariant under a subfamily θ_r, where r runs through a non-trivial sub-semigroup of* $[0,\infty)$.

The σ-algebra of tail events can also be written as

$$\mathfrak{B}^\infty = \bigcap_{t\geqslant 0}\theta_t^{-1}\mathfrak{B},$$

which implies that if B is a tail event, and $t\geqslant 0$, then there exists $B_t\in\mathfrak{B}$ such that $B=\theta_t^{-1}(B_t)$.

In the foliated space M, the Borel subsets which are unions of leaves form a σ-algebra $\mathfrak{F}$, because being a union of leaves is a property which is preserved by the operations of union and complementation. It is clear that each projection π_t is measurable as a map

$$\pi_t : (\Omega(M), \mathfrak{B}^\infty) \to (M, \mathfrak{F})$$

and also that the σ-algebras $\pi_s^{-1}(\mathfrak{F}) = \pi_t^{-1}(\mathfrak{F})$.

Proposition 6.5. *If* $B \subset \Omega(M)$ *is a tail event, then* $P_x[B]$ *is either* 0 *or* 1. *Moreover, the function* $x \mapsto P_x[B]$ *is constant along the leaves.*

Proof. The first sentence is a continuous version of the zero-one law of Kolmogoroff, see [8]. Given such B, there is a sequence A_n of cylinder sets such that $P_x[A_n \triangle B] \to 0$ as $n \to \infty$, where $\triangle$ denotes symmetric difference. If A_n is in $\mathfrak{B}_{t_n}$, then A_n and B are independent, hence $P_x[A_n \cap B] = P_x[A_n]\, P_x[B]$. Upon letting $n \to \infty$, it obtains that $P_x[B]^2 = P_x[B]$.

For each $t > 0$ there exists B_t such that $\chi_B = \chi_{B_t} \circ \theta_t$. If $P_x[B] > 0$ and y is a point in the same leaf L as x, the strong Markov property implies that

$$\begin{aligned} P_x[B] &= E_x[\chi_B] = E_x[\chi_{B_t} \circ \theta_t] \\ &= E_x[E_x[\chi_{B_t} \circ \theta_t \mid \mathfrak{B}_t]] \\ &= \int_L p(x, y; t) P_y[B_t] \cdot \mathrm{vol}(y). \end{aligned}$$

Since, for any $t > 0$, the heat kernel $p(x, y; t)$ is strictly positive with $\int_L p(x, y; t) = 1$, this identity says that if $P_x[B] = 1$ (respectively 0), then also $P_y[B_t] = 1$ (respectively 0) for every other point y in the same leaf as x, and so $P_y[B] = P_x[B]$ for any two points x, y in the same leaf. $\square$

Theorem 6.6. *The harmonic measure* m *on* M *is ergodic if and only if its associated measure* μ *on* $\Omega(M)$ *is ergodic for the shift* θ.

Proof. Let B be a θ_t-invariant Borel subset of $\Omega(M)$. By the above, the probability $P_x[B]$ is either 0 or 1, and the function $x \to P_x[B]$ is constant along the leaves of M. If m is ergodic, then $P_x[B]$ is constant almost everywhere on M. Thus $\mu(B) = \int P_x[B] \cdot m(x)$ is either 0 or 1.

Conversely, if m was not ergodic, then there would be a measurable union of leaves $A \subset M$, with $0 < m(A) < 1$. Let B be the collection of leaf paths ω such that $\omega(0) \in A$. If $\omega(0) \in A$, then $\omega(t) \in A$ for all t; so the set $B = \pi_0^{-1}(A)$ is

θ_t-invariant. Moreover

$$\mu(B) = \int_M \left(\int_{\Omega(M)} \chi_B(\omega) \cdot P_x(\omega) \right) \cdot m(x) = m(A),$$

implying that μ is also not ergodic. $\square$

Corollary 6.7. *If m is ergodic and $t>0$, then the semigroup $\{\theta_{nt} \mid n \geqslant 0\}$, is ergodic on $\Omega(M)$.*

Proof. If B is invariant under the iterates of θ_t, then B is a tail event. $\square$

7. The ergodic theorem and the Kryloff–Bogoliouboff theory

This section contains the ergodic theorem for the diffusion semigroup of a Laplace operator on a foliated space. It turns out that the diffusion semigroup has stronger properties, for it will be shown that the action of D_t in $L^2(m)$ is mixing whenever m is an ergodic probability harmonic measure.

Let M be a foliated space with diffusion semigroup D_t. The statement of the ergodic theorem for D_t requires two preliminary facts. They are contained in Garnett's paper [14], although a slightly simpler proof of the first will be offered.

Proposition 7.1. *If m is a probability harmonic measure on M, then D_t induces a contraction semigroup of operators on the Banach spaces $L^p(m)$, for $1 \leqslant p < \infty$, and $\|D_t\| \leqslant 1$ for $1 \leqslant p \leqslant \infty$.*

Proof. If f is continuous, then

$$\|f\|_1 = \int_M |f|(x) \cdot m(x) \leqslant \|f\|.$$

Moreover,

$$\int |D_t f|(x) \cdot m(x) \leqslant \int D_t |f|(x) \cdot m(x) = \int |f|(x) \cdot m(x),$$

so that D_t is a continuous linear functional on the dense subspace of continuous functions in $L^1(m)$. Therefore it extends to a linear operator D_t on $L^1(m)$ with $\|D_t\| \leqslant 1$. The same works for all other L^p-spaces, $1 < p < \infty$. The case $p = \infty$ is equally easy, because $D_t f(x)$ is represented by a probability measure on M, and m is D_t-invariant.

What this method of extension does not say is how to interpret $D_t f(x)$ when f is only integrable (or positive). With the concepts introduced in the previous section, note that $f \circ \pi_t$ is in $L^1(\mu)$. From the definition of μ, it follows that $f \circ \pi_t$ is

P_x-integrable for almost all $x \in M$. For these x, $D_t f(x) = E_x[f \circ \pi_t]$. The semigroup property follows from the Markov property. See [14,38].

That D_t is a contraction semigroup on the Banach spaces $L^p(m)$ is due to the following. If f is continuous, then

$$||D_t f - D_s f||_1 \leqslant ||D_t f - D_s f||.$$

It follows that the function $t \to D_t f$ of $[0, \infty)$ into $L^1(m)$ is norm continuous on the dense subspace of $L^1(m)$ consisting of continuous functions. A standard argument now shows that it is norm continuous on $L^1(m)$. Given $f \in L^1(m)$ and $\varepsilon > 0$ let f' be continuous with $||f - f'||_1 < \varepsilon$. Then

$$||D_t f - D_s f||_1 \leqslant ||D_t f - D_{s+t} f'||_1 + ||D_t f' - D_s f'||_1 + ||D_t f' - D_s f'||_1,$$

so that $\lim_{t \to s} ||D_t f - D_s f|| \leqslant 2\varepsilon$. $\square$

The proposition shows that D_t is a semigroup of bounded linear operators of $L^1(m)$ with L^1 and L^∞-norm bounded by 1. An application of the operator ergodic theorem will produce a function which is D_t-invariant. It turns out that D_t-invariant functions classes are constant along the leaves.

Proposition 7.2. *Let m be a harmonic probability measure on the foliated space M and let f be a diffusion invariant function which is m-integrable. Then the class of f in $L_1(M, \mu)$ contains a function which is constant along the leaves of M. In fact, invariance for one positive time t implies f is constant on almost every leaf.*

Proof. See Garnett's paper [14], where other related results are also proved. Note also that if f is an excessive function, that is, a measurable function $f : M \to [0, \infty]$ such that $D_t f \leqslant f$ and $\lim_{t \to 0} D_t f = f$, then f is constant almost everywhere. Alternatively, these facts can be proven by considerations in $\Omega(M)$. $\square$

The following ergodic theorem is then obtained.

Theorem 7.3. *Let M be a compact foliated space with a harmonic probability measure m and let f be an m-integrable function on M. Then the time averages*

$$\frac{1}{T} \int_0^T D_t f(x) \cdot dt$$

converge almost everywhere as $T \to \infty$ to an integrable function f^* which is constant along the leaves, and

$$\int_M f(x) \cdot m(x) = \int_M f^*(x) \cdot m(x).$$

If $f \in L^1(m)$, then the time averages $(1/T)\int_0^T D_t f \cdot dt$ converge in $L^1(m)$ to a function class $f^* \in L^1(m)$ which is constant along the leaves, and with $\int_M f \cdot m = \int_M f^* \cdot m$.

This type of continuous ergodic theorem uses Proposition 7.1. It requires some careful considerations not present in the well-known discrete version, but are nevertheless well covered in [11, Chapter VIII].

In Garnett's paper there is a question about the behavior of diffusion, i.e., it is asked whether

$$\lim_{t \to \infty} D_t f(x) = \int_M f \cdot m,$$

holds for almost all $x \in M$ with respect to m, where m is an ergodic probability harmonic measure and f is an integrable function. The following discussion is partly motivated by this question.

Remark. That the semigroup D_t should positively satisfy the question asked by Garnett comes from a study of Orey's work [30], which led to the zero–two theorem of Ornstein–Sucheston [31]. It is noted that such result (Theorem 7.5) is stated by Kaimanovich in [24].

Definition 7.4. A semigroup $\{T_t\}$ of linear operators on a Hilbert space $\mathscr{H}$, with inner product $\langle \, , \, \rangle$, is called mixing if

$$\lim_{t \to \infty} \langle T_t f, g \rangle = \langle f, 1 \rangle \langle 1, g \rangle,$$

for all $f, g \in \mathscr{H}$.

Theorem 7.5. *Let m be an ergodic harmonic measure on M. Then the diffusion semigroup D_t is mixing on $L^2(M)$.*

For $t > 0$ and $\alpha > -t$, let $h_t^\alpha(x)$ be the total variation $h^\alpha(x) = ||D_{t+\alpha}\delta_x - D_t\delta_x||$. For each fixed x, h_t^α is a positive non-increasing function of t, hence the limit

$$h^\alpha(x) = \lim_{t \to \infty} ||D_{t+\alpha}\delta_x - D_t\delta_x||$$

exists. Furthermore, $h^\alpha(x)$ is a measurable function of (x, α).

The following theorem is due to Winkler [37], and is a continuous version of the zero–two theorem of Orstein–Sucheston [31]. The hypotheses of [37] are as follows: (1) if B is a subset of M with $m(B) = 0$, then $D_t\chi_B(x) = 0$ for almost all $x \in M$; and (2) the operator D_t is ergodic for all $t > 0$. The later follows from the similar property in path space proven in Corollary 6.7, while (1) follows immediately from the identity $m(B) = \int_M \chi_B \cdot m$ and D_t invariance of m.

Theorem 7.6. *Let m be an ergodic harmonic measure on M. Then the function h^α is constant almost everywhere. Moreover, either $h^\alpha \equiv 0$ for almost all $\alpha \in \mathbb{R}$ or $h^\alpha \equiv 2$ for almost all $\alpha \in \mathbb{R}$.*

Lemma 7.7. *Let ν and μ be mutually absolutely continuous probability measures on a measure space X. Then the total variation $||\nu - \mu|| < 2$.*

Proof. Let $\nu = f\mu$. The Hahn decomposition of $\nu - \mu$ gives sets A and $X \backslash A$ such that $f \geqslant 1$ on A and $f \leqslant 1$ on $X \backslash A$. Then $(\nu - \mu)^+ = (f-1)\chi_A \mu$ and $(\nu - \mu)^- = (1-f)\chi_{(X \backslash A)}\mu$. The total variation is therefore

$$|\mu|(X) = (\nu - \mu)^+(X) + (\nu - \mu)^-(X),$$

which equals

$$\nu(A) - \mu(A) - \nu(A^c) + \mu(A^c) = 2(\nu(A) - \mu(A)).$$

Since ν and μ are mutually absolutely continuous, the difference $\nu(A) - \mu(A) < 1$. $\square$

If $h^\alpha = 2$, then $h_t^\alpha = 2$ for all t. This implies that there is a null set $N \subset M$ such that for every $x \in M \backslash N$ the measures $D_t \delta_x$ and $D_{t+\alpha}\delta_x$ are mutually singular for all t, in view of Lemma 7.7. But, if $t > \max\{-\alpha, 0\}$, these measures are absolutely continuous; indeed, a subset A on M has positive $D_t\delta_x$ measure if and only if A meets the leaf through x in a set of positive measure, a fact which is independent of the value of $t > 0$.

It follows from Lemma 3 in [37] and the above that $h^\alpha = 0$ almost everywhere on M with respect to M. The following result is then a corollary of this and of the arguments in the zero–two theorem of Ornstein–Sucheston [31].

Corollary 7.8. *Let m be an ergodic harmonic measure on M. Then*

$$\lim_{t \to \infty} ||D_t f||_1 = 0,$$

for every f in $L^1(m)$ with zero mean.

The consequence is the mixing property of the semigroup D_t.

Proof of Theorem 7.5. Let $L_0(m)$ be the subspace of $L^2(m)$ consisting of functions of zero mean. By Schwarz inequality, $|\langle D_t f, g \rangle| \leqslant ||D_t f||_2 ||g||_2$, and so, to exhibit the mixing condition for D_t, it suffices to show that $||D_t f||_2 \to 0$ for every $f \in L_0(m)$.

Let E be the collection of elements $f \in L_0(m)$ such that $||D_t f||_2 \to 0$. Then E is a linear subspace of $L_0(m)$, and is also closed. Indeed, let $f_n \in E$ and $f \in L_0(m)$ be such that $||f_n - f||_2 \to 0$. Then, for every n and t,

$$\begin{aligned} ||D_t f||_2 &\leqslant ||D_t(f_n - f)||_2 + ||D_t f_n||_2 \\ &< ||f_n - f||_2 + ||D_t f_n||_2. \end{aligned}$$

Let $\varepsilon > 0$ and let n be sufficiently large so that $||f_n - f||_2 < \varepsilon$. The displayed inequality and $f_n \in E$ imply that

$$\lim_{t \to \infty} ||D_t f||_2 \leqslant \varepsilon,$$

hence $f \in E$.

If f is a continuous function on M, then

$$||D_t f||_2^2 = \int_M |D_t f|^2 \cdot m \leqslant ||f|| \, ||D_t f||_1,$$

and the last term converges to 0 as $t \to \infty$, by the zero–two theorem (Corollary 7.8). Therefore, the space E contains all such continuous functions in $L_0(m)$. Since these continuous functions are dense in $L_0(m)$, it follows that $E = L_0(m)$. $\square$

Having described the construction of the space $\Omega(M)$ and all the associated paraphernalia, and having stated the form of the ergodic theorem that will be used, the ergodic decomposition of a harmonic probability measure in the style of Kryloff and Bogoliouboff [27] will be presented. The description in [14] uses a different approach based on Yosida [38].

The idea is to study the dynamics of the foliated space M via the semigroup θ acting on the path space $\Omega(M)$. This space $\Omega(M)$ is a complete separable metric space, hence every continuous function on it is Borel measurable. A theorem of Ulam guarantees that a finite Borel measure μ on $\Omega(M)$ is regular.

In order to extend the theory of Kryloff and Bogoliouboff to compact foliated spaces, the definition of quasi-regular points must be done via the path space $\Omega(M)$. The reason is that the function f^* (given by the ergodic theorem) is not canonically associated to f. In fact, if f is continuous, convergence of the averages of f at a point x does not necessarily imply their convergence at other points in the leaf through x. The first part of the discussion that follows will be to resolve the objection just made; once done, the discussion follows [27,29,32].

Definition 7.9. A Borel subset B of M is said to have harmonic measure one if $m(B) = 1$ for every probability harmonic measure m on M.

Let W denote the set of leaf paths ω in $\Omega(M)$ for which the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f\circ\pi_t(\omega)\cdot dt$$

exists for each continuous function f on M.

The set W may, in principle, be larger than the set of quasi-regular points of the semigroup θ on $\Omega(M)$ because, this space being non-compact, a more restrictive definition is required, see [32].

Proposition 7.10. *The set W is a Borel θ-invariant subset of $\Omega(M)$. Moreover, if μ is a probability measure on $\Omega(M)$ induced by a harmonic measure on M, then $\mu(W)=1$.*

Proof. The set W is measurable because it can be written as a countable intersection of measurable sets, each one being the set of points where a limit of a sequence of continuous functions exists.

Every harmonic measure m on M induces a shift invariant measure μ on $\Omega(M)$ with respect to which W has total mass, by the ergodic theorem applied to the shift θ. This same theorem also implies that W is shift invariant, because it is obtained as a countable intersection of such sets. $\square$

Since W is shift invariant, the function

$$x\in M\mapsto P_x[W]$$

is measurable and constant along the leaves of M. Moreover, by the foliated zero–one law of Proposition 6.5, it takes on the values 0 and 1 only. Let Q be the subset of M defined by

$$Q=\{x\in M \mid P_x[W]=1\}.$$

The relation between a harmonic measure m on M and the measure μ that it induces on $\Omega(M)$ implies

$$m(Q)=\int_W P_x[W]\cdot m(x)=\mu(W)=1,$$

which means that the set Q has measure one with respect to every probability harmonic measure m on M. The set Q is called the set of quasi-regular points of M. It is a measurable union of leaves by virtue of the zero–one law (Proposition 6.5), and it has the following fundamental property.

Proposition 7.11. *For every $x \in Q$ and every continuous function f on M, the limit*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T D_t f(x) \cdot dt$$

exists.

Proof. The value $D_t f(x)$ of the diffusion of f at x is the expected value

$$E_x[f \circ \pi_t] = \int_{\Omega(M)} f \circ \pi_t(\omega) \cdot P_x(w).$$

For ω in W, the functions

$$\frac{1}{T} \int_0^T f \circ \pi_t(\omega) \cdot dt$$

are integrable and converge pointwise to an integrable function. If x is such that $P_x[W] = 1$, then the dominated convergence theorem provides the existence of the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T D_t f(x) \cdot dt = P_x \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T f \circ \pi_t(\omega) \cdot dt \right).$$

The interchange of limits is justified by Fubini's theorem, as f is bounded and the measures are probability measures. $\square$

If x is a quasi-regular point, then

$$f \in C(M) \mapsto \lim_{T \to \infty} \frac{1}{T} \int_0^T D_t f(x) \cdot dt$$

is a positive continuous linear functional of norm 1, which takes each constant function to its value. This means that there is a probability measure m_x in M such that

$$f^*(x) = \int_M f(y) \cdot m_x(y).$$

The measure m_x is a probability harmonic measure, called the diffused Dirac measure at x.

From this construction, the following ergodic decomposition is obtained.

Theorem 7.12. *If f is a bounded Borel function on M, then*

$$\int_M f(x) \cdot m(x) = \int_Q \left(\int_M f(y) \cdot m_x(y) \right) \cdot m(x),$$

for every harmonic measure m.

Proof. For any positive number K, the set of measurable functions bounded by K which satisfy the conclusion of the theorem is closed under pointwise limits, and contains all continuous functions on M with supremum norm bounded by K. □

The development of the theory of Kryloff and Bogoliouboff requires the introduction of two further classes of points. A quasi-regular point $x \in Q$ is called transitive, denoted $x \in Q_T$, if m_x is an ergodic measure. It is said to be a point of density, written $x \in Q_D$, if $m_x(U) > 0$ for every open subset U of M containing x. Quasi-regular points which are both points of density and transitive are called regular, and the set of regular points is denoted by R.

Proposition 7.13. *The set Q_D of density points is a Borel saturated set of harmonic measure one.*

Proof. A point $x \in Q$ is a point of density if $f^*(x) > 0$ for every non-negative continuous function on M for which $\{f > 0\}$ is a neighborhood of x.

For each continuous function f on M, the function $f^*(x) = \int_M f \cdot m_x$ is bounded and (Borel) measurable. Moreover, from the definition it follows that f^* is D_t-invariant, and that $(D_t f)^* = f^*$. Furthermore, if f is non-negative and $f^*(x) = 0$, then $f^* = 0$ in the whole leaf through x. Indeed, the expression

$$f^*(x) = D_t f^*(x) = \int_L f^*(y) p(x, y; t) \cdot \mathrm{vol}(y)$$

implies that f^* must be positive on a set of positive measure of the leaf L through x (with respect to the heat kernel density $p(x, y; t)$), hence on the whole leaf for the same reason.

Therefore, if x is a point of density and y is a point in the same leaf which is not a point of density, then there exists a non-negative function f such that $\{f > 0\}$ is a neighborhood of y and $f^*(y) = 0$. Then $D_1 f$ is positive on an open saturated set which is a neighborhood of x, implying, by the previous considerations, that $f^*(x) = 0$, and contradicting the density character of x.

The proof that density points form a Borel set proceeds as follows. For each open set U_n of a countable base for the topology of M, let f_n be a function which is positive on U_n and equal to 0 on its complement. Let B_n denote the set of points of

M where the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T D_t f_n(x)\cdot dt > 0,$$

or $f_n = 0$. Each of these sets is measurable and of harmonic measure one, hence the set $Q \cap (\bigcap_n B_n)$, which is the set of density points, is also measurable and of harmonic measure one. $\square$

Proposition 7.14. *The set Q_T of transitive points is a union of leaves and it has harmonic measure one.*

Proof. If m_x is a harmonic ergodic measure, then the associated measure μ_x in path space is also ergodic. The argument used in [27,29] or [32] applies here as well. $\square$

Corollary 7.15. *The set of regular points R has harmonic measure one. The ergodic decomposition theorem remains true if, in the integral representation, Q is replaced by R, namely for any Borel set $A \subset M$, $m_x(A)$ is Borel measurable on Q and*

$$m(A) = \int_R m_x(A)\cdot m(x)$$

for every harmonic measure.

A Borel set B in M has harmonic measure zero if and only if $m(B) = 0$ for every ergodic harmonic measure m.

By the ergodic theorem, if m is an ergodic measure, then $m_x = m$ for all x except a set of m-measure zero. The set of all quasi-regular points x where $m_x = m$ is called the quasi-ergodic set of m, and the part of this set consisting of regular points is the ergodic set of m. Each of these sets is invariant, and to distinct ergodic measures correspond disjoint quasi-ergodic sets. There is a correspondence between (quasi)-ergodic sets and ergodic measures, and each ergodic measure vanishes outside the corresponding set.

A closed invariant set F is called the center of attraction of M if for every neighborhood V of F the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T D_t\chi_V(x)\cdot dt = 1$$

for every $x \in M$. If F has no closed saturated set which is also a center of attraction, then F is called a minimal center of attraction.

The following result is proven like in [29].

Proposition 7.16. *The closure of the set R of regular points of the compact foliated space M is a minimal center of attraction.*

Returning to the asymptotic behavior of diffusion, the other reason why it was discussed is the following. If f is a continuous function on M, then a necessary condition for solving the equation $\Delta h = f$ is that f be in the kernel of m, i.e.

$$\int_M f(x) \cdot m(x) = 0.$$

The condition however is not sufficient. A sufficient condition can be described as follows. Let $C(f)$ denote the smallest closed D_t-invariant subspace of $C(M)$ containing all functions $D_t f$, that is, the uniform closure of finite linear combinations of functions $D_t f$. Suppose that

$$\sup_{g \in C(f)} \frac{||D_t g||}{||g||} \to 0$$

as $t \to \infty$. Then a solution to $\Delta h = f$ exists. Indeed, the condition implies that $||D_t f|| \leqslant \alpha(t)||f||$ and that $\alpha(st) \leqslant \alpha(s)\alpha(t)$. It follows that $\alpha(t) \leqslant e^{-at}$ for some $a > 0$. This implies that the function $t \to D_t f(x)$ is integrable on $[0, \infty)$, for each x. Then

$$h(x) = \int_0^\infty D_t f(x) \cdot dt$$

is the required function.

As this is a difficult condition to check, it is natural to ask whether the ratio

$$\lambda_1(t) = \sup_f \frac{||D_t f||}{||f||} < 1$$

over all functions f of mean zero, for some $t > 0$. The number $\lambda_1(t)$ may be thought of as the first (non-trivial) eigenvalue of the semigroup D_t acting on $C(M)$, the first (trivial) eigenvalue being $\lambda_0 = 1$, by reasons of compactness. This is know to be true if M is a compact manifold, see [6], and here it will be shown that this in fact is a necessary and sufficient condition. Similar arguments apply in $L^2(m)$.

Theorem 7.17. *Let M be a compact foliated space. Let Δ be the Laplace operator of a metric tensor, and m a harmonic measure. Then $\lambda_1 < 1$ if and only if M has a compact leaf in the support of m.*

Proof. By the Kryloff–Bogoliouboff theory, if there are no compact leaves, it may be assumed that M has a dense leaf and the measure m is ergodic and positive on open sets. By the well-known result of Hector and of Epstein–Millet–Tischler [5], the support of m must contain leaves without holonomy. Let L be one of these leaves, and let $x \in L$. If $B(r)$ is the ball of radius r centered at x in L, the absence of holonomy implies, by the setup of Section 2, that there exists a transversal T through x and an embedding of the product $B(r) \times T$ in M, sending leaves to leaves. Given a positive number K, it is possible to find two disjoint open subsets E and F of T, relatively compact and with disjoint closure, with F containing the point x, such that for any leaf L' the distance between $L' \cap E$ and $L' \cap F$ in L' is larger than K. Since $B(r) \times T$ is a product, a function f with the following properties can easily be constructed. First observe that E can be assumed to be homeomorphic to F because the lamination is minimal, so the transverse structure is almost homogeneous. Then construct f on $B(r) \times E$ so that f is non-negative, has support in $B(r) \times E$, and is equal to 1 in $B(r/2)$. Then take $-f$ on $B(r) \times F$. Let f denote the function so constructed. It is clear that $\|f\| = 1$, and perhaps after a slight modification of f on $B(r) \times F$, it may be assumed that $\int_M f \cdot m = 0$. Moreover, if $t > 0$ and $\varepsilon > 0$ are given numbers, there exists an arbitrarily large number $r > 0$, after which K, E and F can be found, so that $\|D_t f\| \geqslant 1 - \varepsilon$. This is so because $p(x, y, t)$ is concentrated in the ball $B(r)$, i.e., given ε, there exists r so that

$$\int_{B(r)} p(x, y, t) \cdot \mathrm{vol}(y) \geqslant 1 - \varepsilon,$$

and choosing K sufficiently large, the region $B(r)$ in L may be assumed to interfere very little with those regions of L in $B(r) \times F$. $\square$

8. Asymptotic values of cocycles

The purpose of this section is to prove an ergodic theorem for certain type of cocycles, and to describe some consequences. The following data will be fixed through this section. Let M be a compact foliated space with associated path space $\Omega(M)$ and semigroup θ. Let Δ be a Laplace operator on M, m be a (probability) ergodic harmonic measure on M, and μ be the measure that it induces on $\Omega(M)$. Without real loss of generality, it will be assumed that m is not supported in a single compact leaf.

Associated to the Laplace operator Δ there is a metric tensor along the leaves. The induced distance on the leaves will be denoted by d. Since M is compact, each leaf becomes a Riemannian manifold of bounded geometry. The function d can be extended to $M \times M$ by setting $d(x, y) = \infty$ if x and y are in distinct leaves; it is then a lower semicontinuous function [1], hence measurable. The Laplace operator Δ is of the form $\Delta = \Delta_0 + X$, where Δ_0 is the Laplace operator of the metric tensor, and X is a vector field along the leaves. Associated to Δ there is an operator δ which sends one-forms to functions defined by

$$\delta\alpha = \delta_0\alpha + \alpha(X),$$

where δ_0 is the adjoint of the exterior derivative operator d on functions with respect to the metric tensor of Δ. In particular, if f is a function, then $\delta df = \Delta f$.

Definition 8.1. A cocycle (additive functional) on $\Omega(M)$ is a Borel map

$$A : \Omega(M) \times \mathbb{R}_+ \to \mathbb{R}$$

such that there exists a subset Ω_A of $\Omega(M)$ such that $P_x[\Omega_A] = 1$ for all $x \in M$, $\theta_t(\Omega_A) \subset \Omega_A$, for which the following properties hold for all $\omega \in \Omega_A$:

(1) for each t, the map A_t is $\mathfrak{B}_t$-measurable;
(2) the map $t \mapsto A_t(\omega)$ is continuous, $A_0 = 0$; and
(3) $A_{s+t}(\omega) = A_s(\theta_t\omega) + A_t(\omega)$, for all $s, t \in \mathbb{R}_+$.

If the cocycle equation (3) holds for all ω, then A is called a strict cocycle. There are techniques to improve a cocycle to a strict cocycle. If the equality in (3) is replaced by an inequality "$\leqslant$", then A is called a subcocycle.

A (strict) cocycle A on $\Omega(M)$ defines a semiflow τ on $\Omega(M) \times \mathbb{R}_+$ by

$$\tau_s(\omega, t) = (\theta_s\omega, A_s(\omega) + t).$$

Recurrence properties of A are interpreted by similar properties of this dynamical system.

The asymptotic value of a cocycle A_t is the function, if defined,

$$\lim_{t \to \infty} \frac{1}{t} A_t(\omega).$$

For example, if f is a reasonable function on M, then $\int_0^t f(\omega(t)) \cdot dt$ is a cocycle which, by the ergodic theorem, has asymptotic values almost everywhere on $(\Omega(M), \mu)$.

Lemma 8.2. *Let A_t be a cocycle on $\Omega(M)$ such that A_t is μ-integrable for each t. Then the function*

$$t \in [0, \infty) \mapsto \int_{\Omega(M)} A_t \cdot \mu$$

is linear.

Proof. Integrability of the function of paths $A_t(\omega)$ and shift invariance of the measure μ imply that

$$\begin{aligned}\int_{\Omega(M)} A_{s+t}(\omega) \cdot \mu(\omega) &= \int_{\Omega(M)} A_t(\theta_s\omega) \cdot \mu(\omega) + \int_{\Omega(M)} A_s(\omega) \cdot \mu(\omega) \\ &= \int_{\Omega(M)} A_t(\omega) \cdot \mu(\omega) + \int_{\Omega(M)} A_s(\omega) \cdot \mu(\omega)\end{aligned}$$

as desired. $\square$

Two cocycles A and B on $\Omega(M)$ are cohomologous if there is a Borel function $\varphi : \Omega(M) \to \mathbb{R}$ such that

$$A_t(\omega) - B_t(\omega) = \varphi(\omega) - \varphi(\theta_t\omega),$$

for all t and almost all $\omega \in \Omega(M)$. The cocycles A and B are cohomologous in the sense of Liapunoff if there is such φ with the property that

$$\lim_{t \to \infty} \frac{1}{t} |\varphi(\theta_t \omega)| = 0.$$

This cohomology theory preserves asymptotic values.

The cocycles about which something can be said regarding their asymptotic value are those associated to one-forms along the leaves of M. If α is a one-form, and if ω is a path, then

$$A_t(\omega) = \int_{\omega[0,t]} \alpha$$

defines a cocycle on $\Omega(M)$. To make sense of this integral, it needs to be carried out using stochastic calculus, see [22]. But if α is closed, then it can be defined in a straight forward manner. Indeed, since $d\alpha = 0$, the form α is exact when lifted to the universal cover $\tilde{L}$ of L; that is, there is a function f on $\tilde{L}$ such that $df = \alpha$. Then, if ω is a path in L,

$$A_t(\omega) = f(\tilde{\omega}(t)) - f(\tilde{\omega}(0)),$$

where $\tilde{\omega}$ is any lift of ω to $\tilde{L}$. The value $A_t(\omega)$ is independent of the lift $\tilde{\omega}$ and of f.

A comment regarding diffusion in L and on $\tilde{L}$ needs to be made. The Laplace operator Δ on L lifts to L, and commutes with the covering projection $\pi : \tilde{L} \to L$. To the operator Δ is associated the heat kernel $\tilde{p}(\tilde{x}, \tilde{y}; t)$, which is related to $p(x, y; t)$ on L by

$$p(x, y, t) = \sum_{\gamma \in \pi_1 L} \tilde{p}(\tilde{x}, \gamma\tilde{y}; t).$$

If $x \in L$ and $\tilde{x} \in L$ projects to x, there is a canonical identification of paths spaces $\Omega_{\tilde{x}}\tilde{L}$ and $\Omega_x L$, which identifies the respective Wiener measures $\tilde{P}_{\tilde{x}}$ and P_x on them. As a consequence, in the situation above where α is a one-form on L such that $\alpha = df$ on $\tilde{L}$, the following identity holds

$$E_x[f(\omega(t)) - f(\omega(0))] = \tilde{D}_t f(\tilde{x}) - f(\tilde{x}),$$

where $\tilde{D}_t$ is the diffusion operator associated to Δ on $\tilde{L}$, and $\pi(\tilde{x}) = x$. (Here the expression $D_t f(x)$ stands for $E_x[f \circ \pi_t]$.) To avoid complicated notation, it will be written without the tilde. Since geometric properties of L transfer to $\tilde{L}$, this notational change will have no major effect.

In order to state the ergodic theorem for cocycles, several estimates will be required. The first one concerns the integrability of cocycles defined by one-forms.

Definition 8.3. Let L be a complete Riemannian manifold with distance function d. A function f on L such that

$$|f(x) - f(y)| \leqslant \exp(Kd(x,y) + R),$$

for all $x, y \in L$, and some constants K, R is said to be moderate (with constants K, R).

Definition 8.4. The cocycle A_t is called Lipschitz if

$$|A_t(\omega) - A_s(\omega)| \leqslant Kd(\omega(t), \omega(s)),$$

for almost all ω, and it is called moderate if

$$|A_t(\omega) - A_s(\omega)| \leqslant \exp(Kd(\omega(t), \omega(s)) + R).$$

Proposition 8.5. *Let L be a complete, non-compact, Riemannian manifold of bounded geometry with distance function d. Then, given $T \geqslant 0$,*

$$\sup_{x \in L} \int_L d(x,y)^n p(x,y;t) \cdot \mathrm{vol}(y) \leqslant C(L,T),$$

for all $0 \leqslant t \leqslant T$, where $C(L,T)$ is a constant which depends on T and on the geometry of the manifold L, and n is a positive integer. Similarly,

$$\sup_{x \in L} \int_L e^{d(x,y)} p(x,y;t) \cdot \mathrm{vol}(y) \leqslant C(L,T),$$

for some other constant $C(L,T)$.

Proof. Let $V(x,s)$ denote the volume of the ball $B(x,s)$ of radius s about the point x in L. Then the derivative $V'(x,s) = A(x,s)$, for almost all s, where $A(x,s)$ denotes the volume of the boundary $\partial B(x,s)$.

The fact that L has bounded geometry implies that $V(x,s)$ grows at most exponentially. More precisely, there are constants C and k, depending only on the geometry of L, such that $V(x,s) \leqslant ke^{Cs}$, for $s \geqslant 0$ and all $x \in L$. (When L is a leaf of a compact foliated space this easily follows from consideration of a finite covering by flow boxes; in general, it follows from Bishop's comparison theorem.)

The results of [7], quoted in Theorem 1.14, guarantee that the heat kernel of L admits a "normal" estimate of the following form. Given $T > 0$ there is a constant $B = B(L,T)$ which depends on T and on the geometry of L (i.e., on bounds for the curvature and injectivity radius) such that

$$p(x,y;t) \leqslant \frac{B}{t^{d/2}} e^{-d(x,y)^2/16t},$$

for all $0<t\leqslant T$. Then, for $0<t\leqslant T$,

$$\begin{aligned}\int_L e^{d(x,y)}p(x,y;t)\cdot \mathrm{vol}(y) &\leqslant \int_0^\infty \frac{B}{t^{d/2}} e^r e^{-r^2/16t} A(x,r)\cdot dr\\ &= \frac{B}{t^{d/2}}\left[\int_0^\infty \left(\frac{r}{8t}-1\right) e^{r-r^2/16t} V(x,r)\cdot dr\right]\\ &\leqslant \frac{B}{8t^{(d+2)/2}}\int_0^\infty re^{r-r^2/16t} V(x,r)\cdot dr.\end{aligned}$$

The first inequality follows from the normal estimate of the heat kernel. The equality uses integration by parts and the fact that $\lim_{r\to\infty} e^{(C+1)r-(r^2/16t)}=0$. Using the fact that $rV(x,r)\leqslant kre^{Cr}\leqslant ke^{(C+2)r}$, replacing $C+2$ by C and absorbing the constants k and 8 into B, this inequality can be rewritten as

$$\int_L p(x,y;t)\cdot \mathrm{vol}(y)\leqslant \frac{B}{t^{(d+2)/2}}\int_0^\infty e^{Cr-(r^2/16t)}\cdot dr.$$

The change of variables $u=r-8Ct$ transforms the integral on the right-hand side into the integral

$$e^{4C^2t}\int_{-(8Ct)}^\infty e^{-u^2/16t}\cdot du,$$

and a well-known calculation gives

$$\int_{-8Ct}^\infty e^{-u^2/16t}\cdot du\leqslant \int_{-\infty}^\infty e^{-u^2/16t}\cdot du=\sqrt{16\pi t}.$$

After collecting all the inequalities displayed along the proof and renaming constants, the desired result is obtained, namely

$$\int_L e^{d(x,y)}p(x,y;t)\cdot \mathrm{vol}(y)\leqslant Be^{Ct}/t^{(d+1)/2},$$

for all $0<t\leqslant T$ and constants $B=B(L,T)$ and $C=C(L)$. $\square$

Corollary 8.6. *Let f be a Lipschitz (respectively, moderate) function on L of dilatation $\leqslant K$ (respectively, of constants K,R). Then $E_x[f\circ \pi_t]=D_tf(x)$ exists for all $x\in L$ and all $t\geqslant 0$, and*

$$|D_tf(x)-f(x)|\leqslant C(K,L,t).$$

Proof. Suppose that f is Lipschitz. The function f satisfies an inequality of the form $|f(y)|\leqslant Kd(x,y)+|f(x)|$. Thus the proof of Proposition 8.5 implies that f is integrable with respect to $p(x,y;t)\,\mathrm{vol}(y)$,

$$\int_L |f(y)|p(x,y;t)\cdot \mathrm{vol}(y) \leqslant K\cdot C(L,t) + \inf_{x\in L}|f(x)|.$$

With regard to the second claim,

$$\begin{aligned}|f(x)-D_tf(x)| &\leqslant \int_L |f(x)-f(y)|p(x,y;t)\cdot \mathrm{vol}(y)\\ &\leqslant K\int_L d(x,y)p(x,y;t)\cdot \mathrm{vol}(y). \qquad \square\end{aligned}$$

A related calculation will later be required. It also shows that condition $(*)$ in Section 6 guaranteeing continuity of sample paths holds.

Proposition 8.7. *Let $H(\varepsilon,t)$ denote the function*

$$H(\varepsilon,t) = \sup_{x\in L}\int_{L\setminus B(x,\varepsilon)} p(x,y;t)\cdot \mathrm{vol}(y).$$

Then

$$H(\varepsilon,t) \leqslant \frac{B}{t^{(d+1)/2}}e^{-\varepsilon^2/16t},$$

for $0\leqslant t\leqslant 1$ and $16Ct\leqslant\varepsilon$, where C is a constant such that $V(x,s)\leqslant ke^{Cs}$ for every $x\in L$.

Proof. The calculation in Proposition 8.5 (applied to $T=1$) gives, for $0<t\leqslant 1$,

$$\int_{L\setminus B(x,\varepsilon)} p(x,y;t)\cdot \mathrm{vol}(y) \leqslant \frac{B}{8t^{(d+2)/2}}\int_\varepsilon^\infty e^{Cr-(r^2/16t)}\cdot dr. \tag{1}$$

The change of variables $u = r-8Ct$ implies

$$\int_\varepsilon^\infty e^{Cr-(r^2/16t)}\cdot dr = e^{C^2 8t}\int_{\varepsilon-8Ct}^\infty e^{-u^2/16t}\cdot du.$$

Since $0<t\leqslant 1$, the factor $e^{C^2Bt/2}$ can be replaced by a constant which can then be absorbed into the constant B in equation (1).

If also $t\leqslant\varepsilon/16C$, then $\varepsilon-8Ct\geqslant\varepsilon/2$. Hence, for $t\leqslant t_\varepsilon$,

$$\int_{\varepsilon-8Ct}^\infty e^{-u^2/16t}\cdot du \leqslant \int_{\varepsilon/2}^\infty e^{-u^2/16t}\cdot du. \tag{2}$$

The integral $\int_a^\infty e^{-s^2}\,ds$ is known as the error integral, and the following estimate is available [23]. For $a \geqslant 0$,

$$\int_a^\infty e^{-s^2}\,ds \leqslant \frac{2/\sqrt{2}}{a+\sqrt{a^2+1}}\, e^{-a^2}.$$

To apply this to (2), take $s = u/\sqrt{16t}$, use the fact that $t \leqslant \varepsilon/16C$ and suitably build the new constant B out of the old B and C. $\square$

It is evident, on account of the compactness of M, that the bound for H above works for all leaves of M.

Corollary 8.8. *Let $Q(\varepsilon, t) = \sup_{s \leqslant t} H(\varepsilon, s)$. Then Q is an increasing function of t, and*

$$Q(\varepsilon, t) \leqslant \frac{B}{t^{(d+1)/2}}\, e^{-\varepsilon^2/16t},$$

for $0 < t \leqslant \min\{1, \varepsilon/16C, \varepsilon^2/8(d+1)\}$.

Proof. This follows immediately from the previous proposition and an elementary calculation giving the new bound for t. $\square$

Definition 8.9. A closed one-form α on M is said to be moderate if there are constants K, R such that, on the universal cover $\tilde{L}$ of almost every leaf L, the form $\alpha = df$, where f is a moderate function on $\tilde{L}$ of constants K, R.

Corollary 8.10. *Let α be a closed one-form on M. If α is moderate, then its associated cocycle A_t is integrable.*

Proof. The form α is closed on each leaf L, hence it is exact on the universal cover $\tilde{L}$. If f is a function on $\tilde{L}$ such that $df = \alpha$, then f is moderate with constants K, R. Therefore, $A_t(\omega) = f(\omega(t)) - f(\omega(0))$ and

$$\begin{aligned}\int_{\Omega(M)} |A_t(\omega)| \cdot P_x(\omega) &= \int_{\Omega(M)} |f(\omega(t)) - f(\omega(0))| \cdot P_x(\omega) \\ &\leqslant E_x[\exp Kd(\omega(t), \omega(0)) + R],\end{aligned}$$

which is bounded in terms of t and of the geometry of L, which only depends on the compact space M. Therefore, A_t is an integrable function on each $(\Omega(M), P_x)$, and it is integrable on $\Omega(M)$ because $x \mapsto E_x[A_t]$ is a bounded function of x for each t. $\square$

To determine the coefficient of the linear map in Lemma 8.2, the following extension of Dynkin's formula (Theorem 6.2) will be required.

Proposition 8.11. *Let M be a compact foliated space with Laplace operator Δ and diffusion semigroup D_t. Let L be a leaf of M and let f be a function on L such that f, $|df|$ and Δf are moderate (or Lipschitz) functions on L. Then*

$$E_x[f \circ \pi_t] - E_x[f \circ \pi_s] = E_x\left[\int_s^t \Delta f(\omega(r)) \cdot dr\right],$$

for all $0 < s < t$.

Proof. It may be assumed that $f(x) = 0$. The statement is true for functions which are compactly supported in the leaf L. Let $\{\phi_k\}$ be a partition of unity subordinate to a finite covering $\{U_k\}$ of M by flow boxes, so that there are global bounds for $d\phi_k$ and $\Delta\phi_k$. This covering of M induces a countable covering of the leaf L, its elements being the plaques of the intersections $L \cap U_k$. The partition of unity induces one on L which is now subordinate to this covering of L by plaques. Denote the elements of this partition by $\{\phi_k\}$, indexed by the non-negative integers.

For each $k \geqslant 0$, let f_k be the compactly supported function on L defined by

$$f_k = \sum_{i=1}^{k} \phi_i f.$$

Then

$$\Delta f_k = \sum_{i=1}^{k} (\Delta\phi_i) f + (\Delta f)_k + \sum_{i=1}^{k} \langle d\phi_i, df \rangle,$$

from what it follows that

$$|\Delta f_k| \leqslant C_1 |f| + |\Delta f| + C_2 |df|,$$

where C_1 is a bound for $\Delta\phi_i$ and C_2 for $d\phi_i$.

Since the function f_k is compactly supported and of class C^2 on L, it is in the domain of the infinitesimal generator of Δ. Hence

$$E_x[f_k \circ \pi_t] - E_x[f_k \circ \pi_s] = E_x\left[\int_s^t \Delta f_k \circ \pi_r \cdot dr\right].$$

As $k \to \infty$, $f_k \to f$ uniformly on compact sets of L. Moreover, $|f_k| \leqslant |f|$, and since this last function is integrable with respect to P_x, the dominated convergence theorem implies that, as $k \to \infty$, the left-hand side of the above displayed identity converges to $E_x[f \circ \pi_t] - E_x[f \circ \pi_s]$.

The right-hand side will now be examined. As $k \to \infty$, the functions Δf_k converge to Δf uniformly on compact sets, hence

$$\int_s^t \Delta f_k(\omega(r)) \cdot dr \to \int_s^t \Delta f(\omega(r)) \cdot dr$$

for each path ω, since $\omega[s,t]$ is compact. Thus there is pointwise convergence of $\int_s^t \Delta f_k \circ \pi_r \cdot dr$ to $\int_s^t \Delta f \circ \pi_r \cdot dr$ in $\Omega(L)$. Each of the functions

$$\omega \mapsto \int_s^t \Delta f_k(\omega(r)) \cdot dr$$

is integrable with respect to P_x. The convergence is also dominated. Indeed,

$$\begin{aligned}\left|\int_s^t \Delta f_k \circ \pi_r \cdot dr\right| &\leqslant \int_s^t |\Delta f_k \circ \pi_r| \cdot dr \\ &\leqslant \int_s^t C_1 |f \circ \pi_r| \cdot dr + \int_s^t |\Delta f \circ \pi_r| \cdot dr + \int_s^t C_2 |df \circ \pi_r| \cdot dr.\end{aligned}$$

Each of the three terms is treated similarly, using Proposition 8.5 and Corollary 8.6. Assuming that $f(x) = 0$, the first term is bounded above by

$$C_3 \int_s^t \exp(Kd(\bullet, x) + R) \circ \pi_r \cdot dr.$$

It has been previously shown that

$$E_x[\exp d(\bullet, x) \circ \pi_r] \leqslant \frac{B}{r^{(d+1)/2}} e^{Cr},$$

hence, by Fubini's theorem, the function

$$\int_s^t |f \circ \pi_r| \cdot dr$$

is integrable. The statement is thereby proved. □

The following consequence will be relevant to compute the asymptotic value of certain cocycles.

Corollary 8.12. *Let f be a function on L as in Proposition* 8.11, *and suppose moreover that Δf is bounded. Then*

$$\frac{d}{dt} D_t f = D_t \Delta f.$$

Proof. The identity

$$\frac{d}{dt} D_t f = \Delta D_t f$$

holds by partial differential equations theory. It follows from the proposition that, for $t > 0$,

$$\frac{D_{t+s}f(x) - D_t f(x)}{s} = E_x\left[\frac{1}{s}\int_t^{t+s} \Delta f \circ \pi_r \cdot dr\right].$$

Taking the limit as $s \to 0$, it obtains that

$$\frac{d}{dt} D_t f(x) = E_x[\Delta f \circ \pi_t] = D_t \Delta f(x),$$

by the bounded convergence theorem. $\square$

Theorem 8.13. *Let α be a bounded closed one-form on M and suppose that the function $\delta\alpha$ is bounded. The coefficient of the linear map*

$$t \mapsto \int A_t \cdot \mu$$

is the limit

$$\lim_{t \to 0} \frac{1}{t} \int_{\Omega(M)} A_t(\omega) \cdot \mu(\omega) = \int_M \delta\alpha(x) \cdot m(x).$$

It should be clear from the proof that something less than boundedness of the form α and function $\delta\alpha$ is actually required; it suffices that $\delta\alpha$ be of constant sign.

Proof. By definition of the measure μ,

$$\int_{\Omega(M)} A_t(\omega) \cdot \mu(\omega) = \int_M \left(\int_{\Omega(M)} A_t(\omega) \cdot P_x(\omega) \right) \cdot m(x).$$

The integral of A_t with respect to P_x is

$$E_x[A_t] = D_t f(x) - f(x),$$

where f is a function on the universal cover of the leaf L through x which satisfies $df = \alpha$. In view of the boundedness conditions on α, this function f satisfies a Lipschitz condition on L.

As t goes to zero, the function $(D_t f(x) - f(x))/t$ converges pointwise to $\Delta f = \delta\alpha$. Furthermore, $D_t f$ satisfies the differential equation $(d/dt)D_t f = \Delta D_t f$, and $D_t f(x) \to f(x)$ locally in L^1. Therefore,

$$\frac{D_t f(x) - f(x)}{t} = \frac{1}{t}\int_0^t \Delta D_t f(x) \cdot dt.$$

Since $|\alpha|$ and $\delta\alpha$ are bounded, the extension of Dynkin's formula in Proposition 8.11 and Corollary 8.12 applies and gives

$$\frac{E_\bullet[A_{t+s}] - E_\bullet[A_s]}{t} = E_\bullet\left[\frac{1}{t}\int_s^{s+t} \delta\alpha \circ \pi_r \cdot dr\right].$$

As $t \to 0$, the right-hand side converges to $E_x[\delta\alpha \circ \pi_s] = D_s\delta\alpha = D_s\Delta f(x)$. The left-hand side converges to $\Delta D_s f(x)$. This holds for all $s > 0$, hence $\Delta D_t f = D_t \Delta f$ on L, for $s > 0$, and also for $s = 0$ pointwise, by continuity of sample paths. Since $|\alpha|$ is bounded by K on M,

$$\left| E_\bullet\left[\frac{1}{t}\int_0^t \delta\alpha \circ \pi_r \cdot dr\right]\right| \leqslant K,$$

in M, and the bounded convergence theorem gives the result. $\square$

Suppose that A_t is an integrable cocycle on $\Omega(M)$. For each $s \geqslant 0$, the ergodic theorem, applied to the function A_s and the map θ_s, says that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} A_s(\theta_{ks}\omega) = A_s^*(\omega)$$

almost everywhere on $\Omega(M)$ with respect to the invariant measure μ. The function A_s^* is θ_s-invariant and

$$\int_{\Omega(M)} A_s(\omega) \cdot \mu(\omega) = \int_{\Omega(M)} A_s^*(\omega) \cdot \mu(\omega).$$

The cocycle property of A_s implies that

$$\sum_{k=0}^{n-1} A_s(\theta_{ks}\omega) = A_{ns}(\omega) - A_0(\omega)$$

so that

$$\lim_{n\to\infty} \frac{A_{ns}(\omega)}{n} = A_s^*(\omega).$$

The set of ω's for which this identity holds depends on s. The next result improves considerably these observations.

Theorem 8.14. *Let A_t be a Lipschitz cocycle on $\Omega(M)$. The limit*

$$\lim_{t\to\infty} \frac{1}{t} A_t(\omega)$$

exists almost everywhere on $\Omega(M)$.

Proof. Taking $s = 1$ above, the limit

$$\lim_n \frac{1}{n} A_n(\omega)$$

exists and is a constant function of ω in a conull subset of $\Omega(M)$. Let $[t]$ denote the integer part of $t \geqslant 0$. Then, for $t \geqslant 1$,

$$\begin{aligned}\left|\frac{1}{t} A_t - \frac{1}{[t]} A_{[t]}\right| &\leqslant \left|\frac{1}{t} A_t - \frac{1}{t} A_{[t]}\right| + \left|\left(\frac{1}{t} - \frac{1}{[t]}\right) A_{[t]}\right| \\ &\leqslant \frac{1}{[t]} |A_t - A_{[t]}| + \frac{1}{t}\left|\frac{1}{[t]} A_{[t]}\right|.\end{aligned}$$

The last term converges to zero as $t \to \infty$. The Lipschitz condition on A_t gives the inequality (the Lipschitz constant K taken to be $K = 1$ for simplicity)

$$|A_t(\omega) - A_{[t]}(\omega)| \leqslant d(\omega(t), \omega([t])),$$

from what it follows that the set of paths

$$\left\{\omega \,\middle|\, \sup_{n \leqslant t \leqslant n+1} |A_t(\omega) - A_n(\omega)| \geqslant 2n\varepsilon\right\}$$

is contained in the set

$$\left\{\omega \,\middle|\, \sup_{n \leqslant t \leqslant n+1} d(\omega(t), \omega(n)) \geqslant 2n\varepsilon\right\}.$$

Two more facts will be required to estimate the probability of this last event.

Lemma 8.15. *Let $\varepsilon > 0$, let $a < b$ be points in $[0, \infty)$ and let A denote the set of paths*

$$A = \{\omega \in \Omega(M) \mid d(\omega(b), \omega(a)) \geqslant \varepsilon\}.$$

Then, for every $x \in M$,

$$P_x[A] \leqslant Q(\varepsilon, b - a).$$

Proof. The probability of the event A with respect to the measure P_x is computed as follows. Let L denote the leaf containing x. Let $\pi_{ab} = (\pi_b, \pi_a) : \Omega(M) \to M \times M$, and let χ_ε be the characteristic function of

$$\{(x, y) \in M \times M \mid d(x, y) \geqslant \varepsilon\},$$

so that $\chi_A(\omega) = \chi_\varepsilon \circ \pi_{ab}(\omega)$ and $\pi_{0(b-a)} \circ \theta_a = \pi_{ab}$. Then

$$\begin{aligned}
P_x[A] &= E_x[\chi_\varepsilon \circ \pi_{0(b-a)} \circ \theta_a] \\
&= E_x[E_x[\chi_\varepsilon \circ \pi_{0(b-a)} \circ \theta_a \mid \mathfrak{B}_a]] \\
&= E_x[E_\bullet[\chi_\varepsilon \circ \pi_{0(b-a)}] \circ \pi_a] \\
&= \int_L p(x,y;a)\left(\int_{L\setminus B(y,\varepsilon)} p(y,z;b-a)\cdot \mathrm{vol}(z)\right)\cdot \mathrm{vol}(y) \\
&\leqslant \sup_{y\in L} \int_{L\setminus B(y,\varepsilon)} p(y,z;b-a)\cdot \mathrm{vol}(z) \\
&\leqslant H(\varepsilon, b-a),
\end{aligned}$$

where the second equality is by the projection rules of the expected value operation, and the third is by the Markov property. The stated inequality is by the definition of the function Q. $\square$

Lemma 8.16. *Let $\varepsilon > 0$, let $a < b$ be points in $[0,\infty)$, and let A denote the set of paths*

$$A = \left\{\omega \,\middle|\, \sup_{a\leqslant t\leqslant b} d(\omega(a),\omega(t)) \geqslant 2\varepsilon\right\}.$$

Then

$$P_x[A] \leqslant 2Q(\varepsilon, b-a).$$

Proof. Let $\tau(\omega) = \inf\{t\in(a,b] \mid d(\omega(a),\omega(t))\geqslant\varepsilon\}$, the infimum of the empty set being ∞. The function τ is a stopping time. Indeed, first note that the lower semicontinuity of d implies that

$$\{d(\omega(p),\omega(q))\geqslant\varepsilon\} = \bigcap_{n=1}^{\infty}\{d(\omega(p),\omega(q)) > \varepsilon - 1/n\}$$

is a G_δ set in $(\Omega(M),\mathfrak{B}_b)$. For $r>0$, let $\tau_r(\omega) = \inf\{t\in[r,b] \mid d(\omega(a),\omega(t))\geqslant\varepsilon\}$. Let $\mathbb{Q}^+$ denote the positive rationals, and let $s>a$. Then

$$\{\tau_r \leqslant s\} = \bigcap_{n=1}^{\infty}\{d(\omega(a),\omega(t)) > \varepsilon - (1/n) \text{ for some } t\in[r,s]\cap\mathbb{Q}^+\},$$

hence $\{\tau_r\leqslant s\}\in\mathfrak{B}_s$, and so $\{\tau\leqslant s\} = \bigcup_{n=1}\{\tau_{1/n}\leqslant s\}\in\mathfrak{B}_s$, as required.

Continuing with the proof of the lemma, let A_1 be the set

$$A_1 = \{d(\omega(a),\omega(b))\geqslant\varepsilon\}$$

and let A_2 be

$$A_2 = \{\tau < \infty ; d(\omega(\tau(\omega)), \omega(b)) \geqslant \varepsilon\}.$$

Then $A \subset A_1 \cup A_2$, for if $\omega \in A$, then $d(\omega(a), \omega(\tau(\omega))) \geqslant 2\varepsilon$, and the triangle inequality implies that ω must be in either A_1 or A_2.

The probability $P_x[A_1]$ is computed much like that of A in the previous lemma, obtaining

$$P_x[A_1] \leqslant Q(\varepsilon, b-a).$$

To compute $P_x[A_2]$, first use the strong Markov property (Theorem 6.1) and Lemma 8.15 to get

$$E_x[\chi_\varepsilon \circ \pi_{\tau b} | \mathfrak{B}_{\tau+}](\omega) = E_{\pi_\tau(\omega)}[\chi_\varepsilon \circ \pi_{0(b-\tau(\omega))}] \leqslant Q(\varepsilon, b-\tau(\omega)),$$

for P_x-almost all ω in $\{\tau < \infty\}$, and note that if $\tau(\omega) < \infty$, then $\tau(\omega) \in [a, b]$. This, together with Corollary 8.8, implies that

$$\begin{aligned} P_x[A_2] &= P_x[P_x[A_2 \mid \mathfrak{B}_{\tau+}]] \\ &= E_x[E_x[\chi_\varepsilon \circ \pi_{\tau b} | \mathfrak{B}_{\tau+}]; \tau < \infty] \\ &\geqslant Q(\varepsilon, b-a) \quad \square \end{aligned}$$

Resuming the proof of Theorem 8.14, if $n \leqslant t \leqslant n+1$, then the estimates previously carried out show that

$$P_x\left\{\sup_{n \leqslant t \leqslant n+1} d(\omega(t), \omega(n)) \geqslant 2n\varepsilon\right\} \leqslant 2Q(n\varepsilon, 1).$$

Corollary 8.8 shows that

$$Q(n\varepsilon, 1) \leqslant Be^{-(n\varepsilon)^2},$$

whenever $16C \leqslant n\varepsilon$, and $1 \leqslant 16(\varepsilon n)^2/(d+1)$. Once ε is fixed, this holds for all n sufficiently large. Therefore, the series

$$\sum_{n=0}^{\infty} P_x\left\{\sup_{n \leqslant t \leqslant n+1} d(\omega(t), \omega(n)) \geqslant 2n\varepsilon\right\}$$

converges for each $\varepsilon > 0$, and the Cantelli part of the Borel–Cantelli lemma implies that

$$P_x\left(\limsup_n \sup_{n \leqslant t \leqslant n+1} \frac{1}{n} d(\omega(t), \omega(n)) \geqslant 2\varepsilon\right) = 0.$$

Therefore

$$\limsup_{n\to\infty} \sup_{n\leqslant t\leqslant n+1} \frac{1}{n} d(\omega(t), \omega(n)) = 0$$

almost everywhere with respect to P_x. It follows that

$$\limsup_{n\to\infty} \sup_{n\leqslant t\leqslant n+1} \frac{1}{n} |A_t(\omega) - A_n(\omega)| = 0,$$

and the proof is complete. □

Theorem 8.17. *Let M be a compact foliated space and Δ be the Laplace operator of a metric tensor. If m is a ergodic harmonic measure on M which is not totally invariant, then m is not equivalent to a totally invariant measure.*

In fact, the von Neumann algebra of such foliated space (M, m) contains an essential factor of type III, as follows from a theorem of Krieger [26], and from Connes theory of von Neumann algebras of foliations.

Proof. Let η be the modular form of m. Then $\eta = d\log h$, where h are positive harmonic functions on (almost all) the leaves. Because of Harnack's inequalities, if h is harmonic, then $\log h$ is Lipschitz. Therefore, Theorem 8.14 and the discussion prior to it apply to the cocycle A_t associated to η. Thus the limit

$$\lim_{t\to\infty} \frac{1}{t} A_t(\omega) = a, \tag{$*$}$$

for almost all paths ω, and is in fact constant on a conull set, by ergodicity of μ.

That the limit a is $\int_M \delta\eta \cdot m$ does not quite follow as before, because the function $\delta\eta$ may not be bounded. (Although that calculation can be extended to deal with forms α such that $\delta\alpha < 0$.) But in any case,

$$\delta\eta = \Delta \log h = \frac{-|\operatorname{grad} h|^2}{h^2}$$

which is <0 almost everywhere unless h is constant, that is, unless m is totally invariant.

The asymptotic value a of A is

$$a = \lim_{t\to 0} \frac{1}{t} \int_{\Omega(M)} A_t \cdot \mu,$$

and the integrand $\int A_t(\omega) \cdot P_x(\omega)$ is equal to $D_t \log h(x) - \log h(x)$, and

$$D_t \log h(x) - \log h(x) \leqslant \log D_t h(x) - \log h(x) = 0,$$

by concavity of log and D_t-invariance of harmonic functions, with equality at x if and only if h is constant in the leaf through x. Taking Lemma 8.2 into account, it follows that $a<0$.

Suppose now that m is equivalent to a totally invariant measure, not necessarily finite. Then there exists a positive function f such that $\alpha = d\log f$ (such function f may be assumed to be of class C^2 along almost all the leaves). Hence

$$A_t(\omega) = \log \frac{f(\omega(t))}{f(\omega(0))}.$$

Let B be a subset of M of positive measure with the property that $(1/C) \leqslant f \leqslant C$ on B, for some constant $C>1$. An application of Poincaré recurrence to the shift θ, the measure μ and the set of paths $\{\omega(0) \in B\}$ gives that almost every path ω meets B at arbitrarily large times. Hence, for almost all paths,

$$\lim_{\substack{t\to\infty \\ \omega(t)\in B}} \frac{1}{t} \log \frac{f(\omega(t))}{f(\omega(0))} = 0$$

which contradicts $(*)$ above. $\square$

Remark. Kaimanovich points out that the fact that the integral of the cocycle considered in Theorem 8.17 is non-zero is established in his paper [24], and that the value of the integral is connected there with the entropy of the leafwise Brownian motion.

The asymptotic behavior of all the cocycles defined by bounded closed one-forms is governed by the asymptotic behavior of the distance subcocycle, which is defined as follows. Let d denote the distance function on the leaves associated to the Laplace operator Δ. For a path $\omega \in \Omega(M)$, set

$$A_t(\omega) = d(\omega(0), \omega(t)),$$

which is a finite number since ω is in a single leaf. By virtue of the triangle inequality,

$$A_{s+t}(\omega) \leqslant A_t(\omega) + A_s(\theta_t(\omega)),$$

so that A_t is a subcocycle. By the work in this section, it is integrable for each t, and, by the subadditive ergodic theorem of Kingman [25] and Theorem 8.14, the asymptotic value

$$\lim_{t\to\infty} \frac{1}{t} A_t(\omega)$$

exists and is constant on a saturated conull subset of $\Omega(M)$ (with respect to an ergodic harmonic measure).

The geometric content of Theorem 8.17 is that, if there is a harmonic measure which is not totally invariant, then almost all the paths in a conull set of leaves drift off to infinity (in the leaf to which they belong) at definite speed, as it happens in hyperbolic space; and moreover, the holonomy along these paths contracts the transverse measure at a rate inversely proportional to the exponential of the time parameter.

The theorem is a close relative of the vanishing of coefficients for semisimple group representations, and of results of Hurder and Katok to the effect that if the Godbillon–Vey class of a codimension one foliation is non-trivial, then the von Neumann algebra of the foliation contains a factor of type III; see their papers [20,21] and also the paper Connes [9] for more results on this topic. With regard to this connection, there is the well-known unpublished result of Duminy [10] which says that a codimension one foliation with non-vanishing Godbillon–Vey class contains a resilient leaf. The concept of resilient leaf is one usually linked to codimension one foliations, but it can be defined in the general context of foliated spaces (see [5]). Their existence implies also that the geometric entropy of the foliation is positive (see [15]).

These concepts can be somehow related in the sense that it can be shown that the non-vanishing of certain characteristic class implies the existence of a resilient leaf, as it will be presently illustrated. The formulation requires first that the foliated space be a compact manifold M with a foliation $\mathscr{F}$ of class C^1, the leaves being C^2-submanifolds of M. Without loss of generality, $\mathscr{F}$ can be assumed to be transversally oriented. Fix a metric tensor on TM, and let Q be the determinant line bundle of the transverse bundle to $\mathscr{F}$. This line bundle defines a one form α on M as follows. Let $\{U_i\}$ be a covering of M by flow boxes adapted to the foliation, the coordinates of U_i are (x_i, z_i), the z_i being the transverse parameters. Then

$$c_{ij} = \log\det\left(\frac{\partial z_i}{\partial z_j}\right)$$

defines a real valued cocycle on M consisting of functions which are locally constant along the leaves of $U_i \cap U_j$. A choice of a partition of unity $\{\phi_i\}$ subordinate to the covering $\{U_i\}$ associates to the cocycle $\{c_{ij}\}$ a one-form α on M, locally $\alpha \mid U_i = \sum_j c_{ij} d\phi_j$, which restricts to a closed one-form on each leaf. It follows that α defines a cocycle A_t on $\Omega(M)$, which is in fact Lipschitz because M is compact and α is continuous.

This 'transverse volume' form α is a closed relative of the Godbillon–Vey class, and has the following geometric meaning. If σ is a loop in a leaf, then

$$\int_\sigma \alpha = \log Jh_\sigma(0)$$

the logarithm of the Jacobian determinant of the holonomy around σ. In general, if σ is a path on a leaf, then $\int_\sigma \alpha$ measures the distortion of the transverse euclidean volume incurred by the holonomy transformation along σ.

In order to deduce some information from the asymptotic values of the cocycle A_t, some restrictions must be imposed on the transverse structure of the foliated manifold $(M, \mathscr{F})$. The pertinent structure will be that the foliated manifold be transversely quasi-conformal. A C^1-map $f : U \to \mathbb{R}^n$ is K-quasi-conformal if its Jacobian Jf satisfies

$$\frac{1}{K} \sup |f'(x)|^n \leqslant Jf(x) \leqslant K \inf |f'(x)|^n,$$

for every x in U, where

$$\sup |f'(x)| = \sup_{|v|=1} |f'(x)v|^n \quad \text{and} \quad \inf |f'(x)| = \inf_{|v|=1} |f'(x)v|^n.$$

Geometrically, f takes infinitesimal balls to infinitesimal ellipsoids of bounded eccentricity. Analytically, the infinitesimal volume distortion of f is comparable to the infinitesimal length distortion.

Let $(M, \mathscr{F})$ be a C^1-foliated manifold with C^2 leaves, S a complete transversal and $\mathscr{H}$ the holonomy pseudogroup defined by $\mathscr{F}$ on S. The foliation is said to be quasi-conformal if there exists a constant K such that every $f \in \mathscr{H}$ is a K-quasi-conformal map. If M is compact, then being quasi-conformal is independent of the metric of M and of the choice of pseudogroup. Of course, the quasi-conformal constant K may depend on these choices.

Theorem 8.18. *Let $(M, \mathscr{F})$ be a compact foliated manifold. Suppose that $(M, \mathscr{F})$ admits a transverse quasi-conformal structure. If there exists a harmonic measure m such that the integral $\int_M \delta\alpha \cdot m \neq 0$, then there exists a leaf in the support of m containing a loop with contracting holonomy.*

A resilient leaf is a non-proper leaf which has a loop with contracting holonomy. The result then shows that if the support of m consists of non-proper leaves (e.g., a minimal set with more than one leaf) then there exists a resilient leaf. The result may not be optimal, as it appears that there is a closer relationship between the geometric entropy and the asymptotic values of the cocycle of α, and perhaps connections with exponents of convergence, as in the theory of Kleinian groups.

Proof of Theorem 8.18. It may be assumed that the measure m is ergodic and the points of its support are density points. Since α is a closed one-form, Corollary 5.4 implies that m is not totally invariant. Therefore almost all the paths (with respect to the associated measure μ) have the properties discussed above, that is, they drift off to infinity at definite speed, and the distance between endpoints of almost every path segment is proportional to time.

Assume that the integral in the statement is negative, and normalize the metric on Q so that it is -1. If ω is a path in a leaf, let h_s^t denote the holonomy map of the segment $\omega[s,t]$, and Jh_t^s its Jacobian. Then there is a stopping time $T = T(\omega)$ such that for almost every path in almost every leaf the Jacobian $Jh_t^0 \sim e^{-t}$ for $t > T$. The hypothesis of quasi-conformality then implies that holonomy along almost all paths contracts infinitesimal transverse balls. The next step is to integrate this fact.

Almost every leaf in the support of m is recurrent. Let L be one of this leaves, $x_0 \in L$ and ω a path in L starting at x_0 with the property that $Jh_t^0 \sim e^{-t}$ for $t > T$, and also that $d(\omega(s), \omega(t))$ is comparable to $|t-s|$ for $s, t > T$. (Almost all the paths have this property, as previously noted.) Furthermore, after removing an initial segment of ω if necessary, it may be assumed that $T = 0$.

The holonomy along ω will be analyzed in the normal bundle $N(L)$ described in Section 2. (The Hilbert space $\mathbb{E}$ can now be replaced by M.) There is an ε-disc bundle W over L, contained in $N(L)$, which the exponential map sends locally diffeomorphically into M, in such manner that the distance d_Q on the fibers of W is uniformly equivalent to that induced from M. The holonomy of L can be described in this disc bundle. There are several points to notice. From the fact that the ambient space is compact, and that the foliation is of class C^1, hence transversally Lipschitz, it follows that there exists a constant $K \geqslant 1$ such that if c is a segment in L of length $|c|$ starting at x_0, and x is a point in the fiber of W over x_0 where h_c is defined, then $d_Q(h_c(x_0), h_c(x)) \leqslant K^{|c|} d_Q(x_0, x)$. In particular, h_c is defined in the ball $B(x_0, \varepsilon K^{-|c|})$.

Let t be sufficiently large, and h_t the holonomy from $\omega(0)$ to $\omega(t)$. By hypothesis, $Jh_t(x_0) \sim e^{-t}$. Since the segment $\omega[0, t]$ has length comparable to t, and since there are essentially finitely many holonomy transformations of length at most t (by compactness of M), there exists $\delta = \delta(t)$ such that $Jh_t(y) \sim e^{-t/2}$ for all y in a transverse disc $B(x_0, \delta)$. Repeating this argument on consecutive segments $\omega[nt, (n+1)t]$, and taking into consideration the previous paragraph, it follows that there is a transverse disc $B(x_0, r)$ and a path ω starting at x_0 such that the holonomy along ω is defined on $B(x_0, r)$ for all segments $\omega[0, t]$, the Jacobian $Jh_t \sim e^{-t/2}$, and h sends $B(x_0, r)$ into $B(\omega(t), e^{-at} r)$, for some constant $a > 0$ independent of t.

By recurrence, $\liminf d_M(x_0, \omega(t)) = 0$, hence there exists a sufficiently large time t such that, viewing the situation now in M, the map h_t sends the ball $B(x_0, r)$ into $B(x_0, r/2)$. This picture and the information on the behavior of h_s along ω produces a contracting fixed point for h_t, which is induced by the holonomy of a loop in a leaf in the support of m.

If the integral in the statement is positive, then the proof proceeds along similar lines, going from infinity to the present position. $\square$

The last piece of the proof is well known in foliations of codimension one, after Sacksteder [34], but it is perhaps closer to the higher codimensional version presented in [2]. A more precise connection with the situation considered in this last reference can be made. The fact is that for transversely holomorphic foliations of codimension one the transverse volume form is essentially the Chern class of the normal bundle to the foliation, and having available geometric information of the ambient manifold and on the leaves actually permits to know the asymptotic value of its corresponding cocycle. This is the case of laminations by surfaces which are holomorphically embedded in complex projective space in such a way that the foliated structure extends to a holomorphic foliation of a neighborhood. A more detailed study will be presented in [3].

Another example of quite different nature is the affinity cocycle, which exists on every Riemann surface lamination without spherical leaves, and is meaningful when there are conformally euclidean leaves. Its construction and properties are the subject of [4].

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